Jussieu summer school on the Gan-Gross-Prasad conjectures
“Periods and L-functions” (3 talks)

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1 Local periods and the Plancherel formula **2**
1.1 Review of the Ichino-Ikeda conjecture . . . 2
1.2 Relative characters . . . 3
1.3 Abstract Plancherel theorem . . . 3
1.4 Asymptotics and tempered representations (p-adic case): . . . 4
1.5 Plancherel theorem for the group . . . 5
1.6 Plancherel theorem for strongly tempered varieties . . . 7
1.7 Nonvanishing and positivity . . . 8

2 Periods and distinction for spherical varieties **10**
2.1 Where do Euler products come from? . . . 10
2.2 General setting . . . 10
2.3 Euler factorization . . . 12

3 In search of the L-function **13**
3.1 Problems . . . 13
3.2 Cases where the Schwartz space is known . . . 13
3.3 Construction of the basic function . . . 14
3.4 Affine toric varieties . . . 15
3.5 Simple reductive monoids . . . 16
3.6 Unramified calculation for periods . . . 16

*Attention: Formulas may not be very precise. References are also missing.

1 Local periods and the Plancherel formula

Abstract

The global period integrals of the Gan-Gross-Prasad conjectures conjecturally factorize as Euler products of local "periods" given, à la Ichino-Ikeda, by integrals of matrix coefficients. I will explain how to understand these local periods in terms of the local Plancherel formula of the corresponding homogeneous space. This leads to a proof of their positivity when the given representation is distinguished.

1.1 Review of the Ichino-Ikeda conjecture

We will be writing G, H, etc. to denote either G_v, H_v (points over a local field) or $G(\mathbb{A}_k), H(\mathbb{A}_k)$.

The Ichino-Ikeda conjecture can be stated as:

$$\text{Global period} = ? \cdot \textstyle\prod_v \text{local periods.}$$

Here we have a spherical pair $G \supset H$, and by "period" we mean an $H(\mathbb{A}_k)$-, resp. $H(k_v)$-biinvariant pairing:

$$\pi \otimes \tilde{\pi} \to \mathbb{C},$$

where π is an irreducible admissible representation, and $\tilde{\pi}$ denotes its dual.

(We prefer this "bilinear" formulation; notice that a hermitian structure on π is an isomorphism: $\tilde{\pi} \simeq \bar{\pi}$, where $\bar{\pi}$ denotes the complex conjugate.)

The factor $?$ is a "global" rational factor which can seemingly be understood in terms of the relative trace formula. We will not discuss it.

The Euler product should be understood in terms of partial L-values. Namely, at almost every place the evaluation of the local period will give a quotient $L_X(\pi_v)$ of special values of L-functions, and the product should be understood as:

$$L_X^S(\pi) \cdot \prod_{v \in S} (\text{local period at } v),$$

for any large enough finite set S of places.

The global period on an automorphic representation π is given by the integral over $[H]$ (well-defined, say, when π is the space of a cuspidal representation).

The local periods are specified when π_v is *tempered*; for non-tempered representations the conjecture is not so clear. We will discuss the notion of temperedness below.

For a tempered representation π_v the local period is given as:

$$\pi_v \otimes \tilde{\pi}_v \ni \phi_v \otimes \tilde{\phi}_v \mapsto \int_{H_v} \left\langle \pi_v(h)\phi_v, \tilde{\phi}_v \right\rangle dh. \tag{1.1}$$

Remark 1. *On measures: we will be fixing local measures throughout satisfying certain natural compatibility assumptions. Globally, we assume that they are Tamagawa measures.*

The goal of this talk is to analyze and understand this local period.

1.2 Relative characters

There is a dual notion to that of the period, namely the notion of a "relative character". If $X = H\backslash G$ (and assume for now that $X(k_v) = H(k_v)\backslash G(k_v)$ as happens in the Gan-Gross-Prasad periods) then we set $\mathcal{M}(X_v)$ = the space of Schwartz measures on X_v. A *relative character* is any bilinear form on $\mathcal{M}(X)$ which factors through a map:

$$\mathcal{M}(X) \otimes \mathcal{M}(X) \to \pi \otimes \tilde{\pi} \to \mathbb{C}, \tag{1.2}$$

were π is an irreducible admissible representation and the last arrow is the canonical pairing.

(Explain duality.)

Specifying the period is equivalent to specifying the relative character:

Lemma 1.2.1. *There is a canonical bijection between periods and spherical characters for a fixed representation.*

Proof. The passage from periods to relative characters is obvious. For the inverse, notice that any two factorizations of a relative character as in (1.2) have to coincide. Indeed, their difference is a morphism:

$$\mathcal{M}(X) \otimes \mathcal{M}(X) \to \ker \langle \ \rangle \subset \pi \otimes \tilde{\pi},$$

but since π is assumed irreducible the only $G \times G$-invariant subspace of $\ker \langle \ \rangle$ is the zero subspace. □

Given this lemma, we will from now on be working mostly with relative characters instead of periods, keeping in mind that a description for one implies a description for the other.

1.3 Abstract Plancherel theorem

Assume that X carries an invariant measure dx, which we fix, then we get an isomorphism: $\mathcal{M}(X) \simeq \mathcal{S}(X) \subset L^2(X, dx)$, where $\mathcal{S}(X)$ denotes the space of Schwartz functions.

Regarding the unitary representation $\mathcal{H} = L^2(X)$ of G we have, first of all, the abstract Plancherel theorem:

Theorem 1.3.1. *There is an essentially unique decomposition of $\mathcal{H}$ as a Hilbert space direct integral of irreducible unitary representations of G:*

$$\mathcal{H} = \int_{\hat{G}} \mathcal{H}_\pi \mu(\pi).$$

Here $\hat{G}$ is the *unitary dual* of G, $\mathcal{H}_\pi$ is the π*-isotypic space* ($\simeq \pi \hat{\otimes}$ multiplicity) and $\mu(\pi)$ is a measure on $\hat{G}$, the *Plancherel measure*.

There is a lot to be explained about this abstract theorem.

First of all, the notion of *direct integral* of (always separable, here) Hilbert spaces requires:

- a measurable space Z;
- a family of Hilbert spaces $Z \ni z \mapsto \mathcal{H}_z$;
- a collection $\mathcal{C}$ of sections $z \mapsto \eta_z \in \mathcal{H}_z$ which will be called *measurable*, with the properties:
  1. there is a countable subcollection whose specializations span a dense subspace of $\mathcal{H}_z$ for all z;
  2. a section η is measurable iff for every measurable section $\eta' \in \mathcal{C}$ the function $z \mapsto \langle \eta_z, \eta'_z \rangle$ is measurable.
- a measure on Z.

Then the Hilbert space is the completion with respect to the following seminorm of the space of measurable sections on which it is finite:

$$\|\eta\|^2 = \int_Z \|\eta_z\|^2_{\mathcal{H}_z} \mu(z).$$

The unitary dual $\hat{G}$ of G is the set of isomorphism classes of unitary representations, equipped with the Fell topology: A representation π is in the closure of a set $S \subset G$ if its diagonal matrix coefficients can be approximated, uniformly on compacta, by diagonal matrix coefficients of elements of S.

The Fell topology on $\hat{G}$ coincides with the natural topology on the *spectrum of the C^*-algebra of G*. The spectrum of a C^*-algebra is exactly the set of isomorphism classes of its irreducible (Hilbert space) representations, and an equivalent definition of its topology is in terms of closures of ideals.

This topology can be difficult to analyze, is not Hausdorff, but fortunately it coincides with the "obvious" one on the subset of tempered representations – and, conjecturally, on Arthur representations, which are all that should be relevant to automorphic forms. However, in applications, lack of knowledge of these conjectures leads to difficulties.

Now a few buzzwords about this abstract Plancherel theorem: The key point here is:

Theorem 1.3.2. *For any reductive group over a local field or and adele ring, and every irreducible unitary representation π, $\pi(C_c(G)dg)$ is contained in the space of compact operators.*

In the language of C^*-algebras, this says that the C^*-algebra of the group is *liminal/CCR*, and this implies the weaker result that its Von Neumann algebra is of type I. Von Neumann algebras of type I are precisely those whose "factors" are all "irreducible", i.e. such that its representations (on Hilbert spaces) admit a "central decomposition" into "irreducibles tensored by multiplicity".

The C^*-algebra of a group G is the C^*-envelope of the convolution algebra of L^1-measures on G, i.e. the one induced by the norm:

$$\|f\|_{C^*} := \sup_\pi \|\pi(f)\|,$$

where π ranges over all $*$-representations on Hilbert spaces.

The Von Neumann algebra of the group is the Von Neumann envelope of its C^*-algebra (i.e. its weak-* closure in the direct sum over all representations $\mathcal{H}$ of $\mathcal{B}(\mathcal{H})$; notice that $\mathcal{B}(\mathcal{H})$ is dual to the space of trace-class operators, hence the weak-* topology).

1.4 Asymptotics and tempered representations (p-adic case):

For spaces such as those we are considering, there is a much more explicit Plancherel theorem. Moreover, for the group case, the whole unitary dual is irrelevant: the Plancherel measure is concentrated on the *tempered dual*. Although this will not be the case for all homogeneous spaces X that we will encounter, it still seems to be a general principle that the support of Plancherel measure for $L^2(X)$ is concentrated on representation of *Arthur type*. More on that later.

We now concentrate on the group case, $X = H$, $G = H \times H$.

Let $P, P-$ be two opposite parabolics of H, $L = P \cap P^-$ the common Levi, $P = LU$ the Levi decomposition etc. We let H_P be the *boundary degeneration* corresponding to this class of parabolics,

$$H_P \simeq L^{\mathrm{diag}} \backslash (U \backslash H \times U^- \backslash H),$$

and $H \times H$-variety of the same dimension as H.

Remark 2. *For those more familiar with automorphic forms, compare $[G]$ vs. $[G]_P$. The picture to have in mind is that H has a compact-mod-center part, the complement of which is modelled by the "very (anti-)dominant parts" $H_P^{\gg}$ of the various H_P. For the notion of "very (anti-)dominant" keep in mind the example of $N\backslash \mathrm{SL}_2$*

(affine plane minus the origin). Here, the "very anti-dominant" part is a neighborhood of infinity (as opposed to the global case, where it is a neighborhood of the cusp, i.e. of zero).

There is a unique universal, $H \times H$-equivariant map:

$$C^\infty(H) \ni f \mapsto f_P \in C^\infty(H_P)$$

with the property that f and f_P coincide on $H^\gg$, or equivalently on $L^\gg$ embedded "naturally" in both $H^\gg$ and $H_P^\gg$.

Notice that $C^\infty(H_P) = I_{P\times P^-}^{H\times H} C^\infty(L)$. We always use *normalized* induction. In particular, we have an action of $\mathcal{Z}(L)$. Due to our normalization of induction, a $\mathcal{Z}(L)$-eigenfunction with unitary eigencharacter on H_P restricts to a multiple of a character satisfying $|\chi| = \delta_P^{\frac{1}{2}}$ on $\mathcal{Z}(L)$.

Fact. *If π is an admissible G-representation and $\pi \to C^\infty(H_P)$, then the image of π is $\mathcal{Z}(L)$-finite.*

This means that $\mathcal{Z}(L)$ will act with generalized eigencharacters. These are the *exponents* of π (or, more precisely, of the morphism into $C^\infty(H_P)$. In particular, for an admissible representation τ of H the exponents of the composition:

$$\tau \otimes \tilde{\tau} \to C^\infty(H) \to C^\infty(H_P)$$

(where the first map is the matrix coefficient) are called the *exponents of* τ.

A function is called *tempered* if for every class P of parabolics as above, it is bounded on $L^\gg$ by a function with *unitary generalized exponents*. An irreducible representation τ of H is called tempered if its matrix coefficients are tempered. The Harish-Chandra Ξ-function is a specific, positive matrix coefficient of a specific tempered representation with *trivial* exponents in all directions. Hence, up to logarithmic terms, it bounds every tempered function.

1.5 Plancherel theorem for the group

We start from the case of $X = H$. In this case, relative characters are characters and they have a *canonical* normalization:

Lemma 1.5.1. *Let $\pi \simeq \tau \otimes \tilde{\tau}$. The following correspond to each other under the obvious duality:*

- *the matrix coefficient pairing: $\tau \otimes \tilde{\tau} \to C^\infty(H)$;*
- *the character of τ: $\mathcal{M}(H) \ni f \mapsto \pi(f) \in HS(\tau) \simeq \tau \otimes \tilde{\tau} \to \mathbb{C}$.*

Doubling the variables and choosing a Haar measure on H, we get canonical functionals:

$$\mathcal{M}(H) \otimes \mathcal{M}(H) \to \pi \otimes \tilde{\pi} \to \mathbb{C},$$

the *characters*.

(Indeed, the above functional applied to $\mu_1 \otimes \mu_2$ is equal to $\Theta_\pi(\mu_1 \star \mu_2^\vee)$.)

Central decomposition

We have an abelian Plancherel decomposition:

$$L^2(H_P) = \mathrm{Ind}_{P\times P^-}^{H\times H} L^2(L) = \mathrm{Ind}_{P\times P^-}^{H\times H} \left(\int_{\widehat{\mathcal{Z}(L)}} L^2(L/\mathcal{Z}(L), \chi) d\chi \right).$$

A representation σ of L such that $\sigma \otimes \tilde{\sigma} \hookrightarrow L^2(L/\mathcal{Z}(L), \omega_\sigma)$ is called a *discrete series* for L. Those define a direct summand $L^2(L)_{\text{disc}}$ of $L^2(L)$.

Thus, it is easy to relate the Plancherel decomposition of $L^2(H_P)$ to that of L. How can we relate the Plancherel decomposition of $L^2(H)$ to that of L?

The Plancherel measure

Fixing a measure on H, since we have introduced canonical characters we get a canonical measure on $\hat{G}$ satisfying the Plancherel formula:

$$\|\Phi\|^2_{L^2(H)} = \int_{\hat{G}} \|\Phi dh\|^2_{HS(\pi)} \mu_H(\pi).$$

For discrete-mod-center series, this is Haar measure times the *formal dimension/degree*. It is called this way because for (finite-dimensional) unitary representations of compact groups it is precisely equal to the dimension when the measure of the group is normalized to be 1.

For a general family of tempered representations $\pi_\chi = I_P^G(\sigma \otimes \chi)$, where χ varies over unramified unitary characters of the Levi, what the Plancherel measure is a multiple of a fixed Haar measure on these characters by factors involving:

- the formal degree of σ;
- a finite group of "automorphisms" of the representation (σ could be isomorphic to $\sigma \otimes \chi$; and for $w \in W(L)$ we generically have $I_P^G(\sigma) \simeq I_P^G({}^w\sigma)$ (however, the right measure to fix is such that "locally" it is equal to a fixed measure on the group of unramified characters then one does not need to mention this finite group);
- the scalar $j(\sigma)$ by which the standard intertwining operator: $I_P^G(\sigma) \to I_{P^-}^G(\sigma)$ acts on the standard hermitian inner product on each of these spaces.

An important remark: $j(\sigma)^{-1}$ is essentially, conjecturally at least,[1] a γ-factor:

$$|\gamma(0, \sigma, \check{u}_P, \psi)|,$$

and hence depends only on the L-packet of σ, not the representation itself. The conjecture of Hiraga-Ichino-Ikeda on formal degrees predicts that this is the case also for the formal degree, up to a rational factor:

$$d(\sigma) = \frac{\langle 1, \sigma \rangle}{|\pi_0(\mathcal{Z}_{\check{L}'}(\varphi))|} \cdot |\gamma(0, \sigma, \mathrm{Ad}, \psi)|,$$

where L' is the quotient of L by the split component of its center.

Hence, up to the local Langlands conjectures and this rational constant, we can define a canonical measure $\mu_{{}^LH}$ on the set of (tempered) Langlands parameters. (For unramified packets or for classical groups there is no ambiguity about this rational constant.)

Example 1.5.2. (unramified Plancherel measure, split case:)

$$\propto \prod_{\alpha \in \Phi} \frac{1 - e^{\check{\alpha}}}{1 - q^{-1} e^{\check{\alpha}}} (\chi) d\chi.$$

[1] Shahidi has proven it for generic representations, and has reduced it for general tempered representations to two standard conjectures on tempered L-packets: stability and genericity.

1.6 Plancherel theorem for strongly tempered varieties

Let X be strongly tempered

Proposition 1.6.1. *The space $L^2(X)$ admits a Plancherel decomposition:*

$$\langle \Phi_1, \Phi_2 \rangle_{L^2(X)} = \int_{\hat{G}} \langle \Phi_1, \Phi_2 \rangle_\pi \, \mu_G(\pi), \tag{1.3}$$

where μ_G is the canonical Plancherel measure on G. Here $\langle \Phi_1, \Phi_2 \rangle_\pi$ denotes the adjoint of the map:

$$\pi \otimes \bar{\pi} \to C^\infty(X) \otimes C^\infty(X)$$

given by:

$$v_1 \otimes \bar{v}_2 \mapsto \int_H \langle \pi(hx_1)v_1, \pi(x_2)v_2 \rangle \, dh,$$

composed with the unitary pairing on π, i.e.:

$$\langle \; , \; \rangle_\pi : C_c^\infty(X) \otimes C_c^\infty(X) \to \pi \otimes \bar{\pi} \to \mathbb{C}.$$

(All statements up to suitable choices of measures on the spaces.)

Proof. Let π be a unitary representation (endowed with a invariant Hilbert norm, hence with a fixed isomorphism: $\tilde{\pi} \simeq \bar{\pi}$) and let $i_\pi : C_c^\infty(G) \to \bar{\pi} \otimes \pi$ denote the dual of matrix coefficient. (If we identify $\bar{\pi} \otimes \pi$ with a subspace of $\mathrm{End}(\pi)$, the morphism i_π simply maps $f \in C_c^\infty(G)$ to $\pi(f)$.)

The Plancherel formula on G can be written as:

$$\langle f_1, f_2 \rangle_{L^2(G)} = \int \langle i_\pi(f_1), i_\pi(f_2) \rangle_{HS} \, \mu_G(\pi)$$

where $\langle \; , \; \rangle_{HS}$ denotes the Hilbert-Schmidt hermitian form on $\bar{\pi} \otimes \pi \subset \mathrm{End}(\pi)$.

Let $\Phi_i(x) = \int_H f_i(hx)dh \in C_c^\infty(X)$, $f_i \in C_c^\infty(X)$ $(i = 1, 2)$. Then:

$$\langle \Phi_1, \Phi_2 \rangle_{L^2(X)} = \int_G \int_H f_1(hg)\bar{f}_2(g) dh dg = \int_H \langle \mathcal{L}_{h^{-1}}(f_1), f_2 \rangle_{L^2(G)} \, dh$$

where $\mathcal{L}_\bullet$ denotes the left regular representation of G. Keeping in mind that $i_\pi(\mathcal{L}_{h^{-1}f}) = \pi(h^{-1}) i_\pi(f)$ we get:

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{L^2(X)} &= \int_H \int_\pi \langle \pi(h) i_\pi(f_1), i_\pi(f_2) \rangle_{HS} \, \mu(\pi) dh = \\ &= \int_\pi \int_H \langle \pi(h) i_\pi(f_1), i_\pi(f_2) \rangle_{HS} \, \mu(\pi) dh = \\ &= \int_\pi \langle \Phi_1, \Phi_2 \rangle_\pi \, \mu(\pi). \end{aligned}$$

Notice that at all stages these integrals are *absolutely convergent*, justifying our application of Fubini. □

The argument applies also to the Fourier-Jacobi case. For now, we restrict to $G(V) \times G(V)$, where $G(V) =$ unitary or symplectic group, $(\omega, \mathcal{H}_\omega)$ "its" Weil representation (really, need double covers in the symplectic case). We let H be the diagonal copy and consider the unitarily induced representation $L^2(X, \omega) = I_H^G(\omega)$. We have the analogous proposition, in terms of the local periods considered by Hang Xue:

Proposition 1.6.2. *The space $I_P^G(\omega)$ admits a Plancherel decomposition:*

$$\langle\Phi_1,\Phi_2\rangle = \int_{\hat{G}} \langle\Phi_1,\Phi_2\rangle_\pi \, \mu_G(\pi), \tag{1.4}$$

where μ_G is the canonical Plancherel measure on G. Here $\langle\Phi_1,\Phi_2\rangle_\pi$ denotes the adjoint of the map:

$$\pi\otimes\bar{\pi} \to C^\infty(X,\omega)\otimes C^\infty(X,\omega)$$

given by:

$$v_1\otimes\bar{v}_2\otimes f_1\otimes\bar{f}_2 \mapsto \int_H \langle\pi(hx_1)v_1,\pi(x_2)v_2\rangle \langle\omega(h)f_1,f_2\rangle \, dh,$$

($f_1, f_2 \in \mathcal{H}_\omega$), composed with the unitary pairing on π, i.e.:

$$\langle\ ,\ \rangle_\pi : C_c^\infty(X,\omega)\otimes C_c^\infty(X,\omega) \to \pi\otimes\bar{\pi}\to\mathbb{C}.$$

The proof is the same.

1.7 Nonvanishing and positivity

Theorem 1.7.1. *For any tempered representation π, the hermitian form:*

$$\phi_1\otimes\bar{\phi}_2 \mapsto \int_H \left\langle\pi(h)\phi_1,\bar{\phi}_2\right\rangle dh$$

is positive semi-definite.

For a "standard" tempered representation (i.e. unitary induction from discrete series, which is not necessarily irreducible)[2] *the form is nonzero iff $\mathrm{Hom}_H(\pi,\mathbb{C}) \neq 0$.*

Proof. Positivity: True for almost every π by positivity of the L^2-inner product. Follows for every by continuity.

Discrete series: Nonvanishing is easier to prove for a discrete series. The theory of asymptotics implies that for a discrete series π, any morphism $\pi \hookrightarrow C^\infty(X)$ has image in L^2 (mod center). Thus, it has to contribute nontrivially to the Plancherel formula.

Continuous spectrum: Again by the theory of asymptotics, any $\pi \hookrightarrow C^\infty(X)$ has tempered image, and this allows one to show that it is in the weak closure of $L^2(X)$. Hence, π belongs to a unitarily induced discrete series $I_P^G(\sigma)$, and the Plancherel measure of the family $\omega \mapsto I_P^G(\sigma\otimes\omega)$ is nonzero on any neighborhood of the trivial character of P.

The idea behind the following argument is this: As we have seen, only the product of a Plancherel measure with a relative character is well-defined. Given a relative character, the Plancherel measure depends on its asymptotic behavior. Now, fixing the Plancherel measure for the group allows us to compare our relative characters with characters/matrix coefficients on the group, which of course are nonzero.

[2] A tweak of this argument can actually show, under the assumption of multiplicity one for a tempered representation in general position, that there is multiplicity one for every "standard" tempered representation – hence the theorem holds for any irreducible tempered. (Thanks to R. Beuzart-Plessis for pointing this out.)

To formulate a brief argument in this direction, let us restrict ourselves to the multiplicity-free case, and view the Plancherel formula (with fixed, G-Plancherel measure) as an association of morphisms:

$$\hat{G} \ni \pi \supset \pi^\infty \xrightarrow{l_\pi} C^\infty(X)$$

(defined almost everywhere, up to a scalar of absolute value 1), with the property that for every smooth section:

$$v : \pi \mapsto v_\pi \in \pi^\infty$$

the function:

$$\Phi_v := \int l_\pi(v_\pi)\mu(\pi) \tag{1.5}$$

has norm:

$$\|\Phi_v\|^2 = \int \|v_\pi\|_\pi^2 \mu(\pi). \tag{1.6}$$

(Of course, this formula holds more generally by the definition of direct integral of Hilbert spaces, but we chose our sections to be smooth in order for the integral (1.5) to make sense pointwise, as well.)

One should view (1.6) as a *lower bound* for the norm of Φ_v: it says that the norm is bounded below by our choice of section and the Plancherel measure. We seek an upper bound of the form:

$$\int \|l_\pi\|^2 \cdot \|v_\pi\|_\pi^2 \mu(\pi),$$

where $\|l_\pi\|$ is a suitably defined "norm" on l_π. In other words, we wish to show that the actual function (1.5) will be much smaller than predicted if the l_π's were allowed to be zero.

Consider the map $x : G \ni g \mapsto Hg \in H\backslash G = X$ which allows us to think of functions on X as left-H-invariant functions on G. Smoothening on the left by some compact open subgroup K', l_π becomes a morphism:

$$K' \star_L l_\pi : \pi \to C^\infty(G),$$

where G acts on the right hand side by right multiplication. This is nothing else than the application of the matrix coefficient map:

$$\tilde{\pi} \otimes \pi \ni K' \star l'_\pi \otimes \bullet \to C^\infty(G),$$

where l'_π is the H-invariant functional on π corresponding by Frobenius reciprocity to l_π.

Fact: If our section $\pi \mapsto v_\pi$ is K-invariant, for some K, we can choose K' so that on an open subset G^+ which surjects to X we have:

$$\left\langle K' \star l'_\pi, \pi(g) v_\pi \right\rangle = l_\pi(v_\pi)(x(g)).$$

Thus, if we set $f_v(g) = \int \langle K' \star l'_\pi, \pi(g)v_\pi\rangle \, d\pi$, we have:

$$f_v(g) = \Phi_v(x(g))$$

for $g \in G^+$. There is also a comparison of volumes between G^+ and X, which allows us to make an estimate:

$$\|\Phi_v\|^2 \ll \|f_v\|^2,$$

with the implicit constant being independent of v. The right hand side is, in terms of the Plancherel formula for G, equal to:

$$\int \|K' \star l'_\pi\|^2 \cdot \|v_\pi\|_\pi^2 \mu(\pi),$$

which is the estimate that we needed.

□

2 Periods and distinction for spherical varieties

Abstract

The conjectures of Gan-Gross-Prasad fit into the general framework of spherical varieties, and Jacquet's concept of "distinction". I will explain this framework, including, if time permits, methods more general than "periods" such as the Rankin-Selberg method and a speculative attempt for a vast generalization, including Ng's recent foray into reductive monoids. *Remark: there was not enough time to cover this material, and the last part has been moved to the next lecture.*

2.1 Where do Euler products come from?

In the first lecture we discussed the local factors of Ichino-Ikeda (and their generalizations) for the Gan-Gross-Prasad conjecture. The second lecture will describe a general setting from which Euler products arise, and will be highly speculative. In the third lecture, we will discuss the relation of local factors with L-functions.

Multiplicity-one condition. Not quite necessary, answer should come from: RTF.

Clarify that there is a restricted class of examples that are better understood, and a wider class that has not been understood yet:

- situations involving induction from a character of a reductive subgroup;
- more general situations, involving induction of specific, typically "small" representations.

Example 2.1.1. Let U be the unitary group of a skew-hermitian form. The diagonal $U \hookrightarrow U \times U \times U$ is not spherical (except in dimension 2), so we shouldn't expect Euler factorization of the period on three automorphic representations $\pi_1 \otimes \pi_2 \otimes \pi_3$.

However, if we take π_3 to be the Weil representation ω, this is part of the GGP conjectures, and every generic (Vogan) L-packet of $U \times U$ contains a distinguished element. Thus, to the space $\mathrm{Ind}_U^{U\times U}$ we should associate the L-group of $U \times U$.

Finally, if instead of the Weil representation we induce the trivial character, then we are in the group case and only represetations of the form $\tau \otimes \tilde{\tau}$ should be distinguished. (The L-group of this space is the L-group of U under the twisted-diagonal embedding into the L-group of G – twisted by the Chevalley involution that takes the Langlands parameter of a represenation to that of its dual.)

2.2 General setting

Let $X^\bullet = H\backslash G$ be a homogeneous spherical variety. For homogeneous varieties, "spherical" means that the Borel subgroup of G (over the algebraic closure) acts with a Zariski dense (open) orbit. This includes symmetric spaces, the Gross-Prasad cases, flag varieties etc. We will implicitly make some extra (unstated) assumptions on X: multiplicity-one is enough, but one can relax this.

We denote by $\mathcal{S}(X^\bullet)$ the usual Schwartz space of rapidly decaying functions on $X^\bullet$, either locally or globally:

$$\mathcal{S}(X^\bullet(\mathbb{A}_k)) = \sideset{}{'}\bigotimes_v \mathcal{S}(X_v^\bullet),$$

where the restricted tensor product is taken with respect to the characteristic function of $X(\mathfrak{o}_v)$.

Any morphism: $\pi \xrightarrow{H} \mathbb{C}$ gives, by dualization, a morphism:

$$\mathcal{S}(X^\bullet) \to \tilde{\pi},$$

or equivalently a pairing $\mathcal{S}(X^\bullet) \otimes \pi \to \mathbb{C}$.

Explication: If the morphism is given by the automorphic period integral over $[H]$, then we get a pairing:

$$\mathcal{P}_{X^\bullet} : \mathcal{S}(X^\bullet(\mathbb{A}_k)) \otimes \pi \ni \Phi \otimes \phi \mapsto \int_{[G]} \Sigma\Phi \cdot \phi \in \mathbb{C},$$

where $\Sigma\Phi(g) := \sum_{\gamma \in X^\bullet(k)} \Phi(\gamma g) \in C^\infty([G])$.

Notice that, at least when $X^\bullet$ is quasiaffine, the sum converges because $X^\bullet(k)$ is discrete and Φ is of rapid decay.

Remark 3. *The* relative trace formula *for a pair $X_1^\bullet \times X_2^\bullet/G$ is "defined" (up to analytic difficulties) as the G^{diag}-invariant functional:*

$$\Phi_1 \otimes \Phi_2 \mapsto \int_{[G]} \Sigma\Phi_1 \cdot \Sigma\Phi_2$$

on $\mathcal{S}(X_1(\mathbb{A}_k)) \otimes \mathcal{S}(X_2(\mathbb{A}_k))$. Its spectral decomposition is an integral over automorphic representations π of the above morphisms composed with inner product on $\pi \otimes \tilde{\pi}$:

$$\begin{array}{ccccc} \mathcal{S}(X_1(\mathbb{A}_k)) & & & & \mathcal{S}(X_2(\mathbb{A}_k)) \\ \downarrow & & & & \downarrow \\ \tilde{\pi} & & \otimes & & \pi \\ & & \downarrow & & \\ & & \mathbb{C} & & \end{array}$$

Ideological thesis: One should choose an *affine*[1] embedding X on $X^\bullet$, replace $\mathcal{S}(X^\bullet)$ by a suitable Schwartz space $\mathcal{S}(X)$, and repeat the above construction with these new data, obtaining a pairing:

$$\mathcal{P}_X : \mathcal{S}(X(\mathbb{A}_k)) \otimes \pi \to \mathbb{C}. \tag{2.1}$$

Remark 4. *The operator $\Phi \mapsto \Sigma\Phi$ is still defined by summation over $X^\bullet(k)$, not $X(k)$, because as we will see, in the presence of singularities the elements of $\mathcal{S}(X)$ may not extend as functions to the whole space X.*

Example 2.2.1. Both $X^\bullet = \mathrm{GL}_n$ and $X = \mathrm{Mat}_n$ are OK (as $G = \mathrm{GL}_n \times \mathrm{GL}_n$-spaces) because they are both affine. However, the usual Schwartz spaces $\mathcal{S}(X^\bullet(\mathbb{A}_k))$ and $\mathcal{S}(X(\mathbb{A}_k))$ give different answers: The former is dual to $\mathrm{GL}_n^{\mathrm{diag}}$-period integrals (calculating the inner product of two vectors in the automorphic representation, which is Eulerian but at almost every place equal to 1), while $\mathcal{P}_X$ applied to $\Phi \in \mathcal{S}(X(\mathbb{A}_k))$ and two automorphic forms $\phi_1 \in \pi, \phi_2 \in \tilde{\pi}$ is equal to the Godement-Jacquet integral:

$$\int_{\mathrm{GL}_n(\mathbb{A}_k)} \Phi(g) \langle \pi(g)\phi_1, \phi_2 \rangle \, dg,$$

which is Eulerian and equal to a value of the standard L-function of π at almost every place.

Notice also that the continuous parameter of the Godement-Jacquet integral is hidden in the choice of π, which we can vary by $|\det|^s$. The integral is absolutely convergent only under a condition on the central character of π, which we can write as $\Re(\pi) \gg 0$. (This notion of $\Re(\pi)$ can be made rigorous in terms of X.) This will also be the case, in general, for most of the integrals that we will consider.

[1] Throughout: normal. (This is a requirement for X to be called "spherical".)

2.3 Euler factorization

The Euler factorization of $\mathcal{P}_X$, or rather of the corresponding relative character on $\mathcal{S}(X(\mathbb{A}_k)) \otimes \mathcal{S}(X(\mathbb{A}_k))$ which we will denote $|\mathcal{P}_X|^2$ (s. the first lecture) only depends on $X^\bullet$, and hence can be described without reference to the specific embedding X.

Recall that in the Gan-Gross-Prasad cases we had the conjecture, at least for tempered representations:

$$|\mathcal{P}_X|^2 =? \prod_v J_{\pi_v},$$

where $?$ was a small rational factor, the Euler product was understood using partial L-values and J_{π_v} was characterized in terms of its appearance in the Plancherel formula for $L^2(X_v^\bullet)$ *with Plancherel measure* $\mu_{^LG}$. (We think here of the G-Plancherel measure as a measure on Langlands parameters, by the comment on its conjectural independence from the choice of element in the L-packet – up to a small rational factor at a finite number of places.)

In general, we cannot expect $L^2(X^\bullet)$ to be absolutely continuous with respect to G-Plancherel measure. (Example: the group case $X = H, G = H \times H$.) Before we formulate a global conjecure on Euler factorization, we need some "relative local Langlands conjecture" on the Plancherel measure for $L^2(X)$ (over a local field F):

Conjecture 1 (S.- Venkatesh). *There is a direct integral decomposition:*

$$L^2(X) = \int \mathcal{H}_\phi \mu_{^LG_X}(\phi),$$

where $\mu_{^LG_X}$ *is the Plancherel measure on the set of Langlands parameters into the* L*-group of* X*, and* $\mathcal{H}_\phi$ *is a finite direct sum of representations in the Arthur packet with parameter:*

$$\mathcal{W}_F' \times \mathrm{SL}_2 \xrightarrow{\phi\times\mathrm{Id}} {}^LG_X \times \mathrm{SL}_2 \xrightarrow{(2.2)} {}^LG.$$

The L-group of X has for now been defined in generality only in the split case, together with the canonical map used above:

$$^LG_X \times \mathrm{SL}_2 \to {}^LG. \tag{2.2}$$

It is closely related to the one defined by Gaitsgory and Nadler using Tannakian formalism.

Example 2.3.1. For $X = G\backslash G$ the L-group is trivial (equipped with the obvious, trivial map to ^LG, but the map on SL_2 is principal into the connected dual group $\check{G}$, so we get the Arthur parameter of the trivial representation.

For $X = \mathrm{Sp}_{2n}\backslash \mathrm{GL}_{2n}$ we get for connected components: $\check{G}_X = \mathrm{GL}_n$, $\check{G}_X \times \mathrm{SL}_2 \to \check{G}$ the tensor product representation of GL_n and SL_2 to GL_{2n}.

Fixing now the canonical Plancherel measure $\mu_{^LG_X}$ we get relative characters J_{π_v} on $\mathcal{S}(X_v) \otimes \mathcal{S}(X_v)$, and we can state the global conjecture, assuming the notion of global Arthur parameter $\mathcal{L}_k \times \mathrm{SL}_2 \to {}^LG$:

Conjecture 2 (S.-Venkatesh). *Assume that* π *is an automorphic representation which admits global Arthur parameter factoring as:*

$$\mathcal{L}_k \times \mathrm{SL}_2 \xrightarrow{\phi\times\mathrm{Id}} {}^L\check{G}_X \times \mathrm{SL}_2 \to {}^LG,$$

then $|\mathcal{P}_X|^2$ *factorizes as:*

$$? \cdot \prod_v J_{\pi_v},$$

with the same conventions as above on $?$ *and the Euler product.*

This conjecture should be taken with a grain of salt, as its proper formulation seems to involve the relative trace formula and one may need to formulate it in terms of Vogan-Arthur packets, not individual representations in them.

3 In search of the L-function

Abstract

There is very little understanding about the (special value of an) L-function associated to each period. In anticipation of a nicer answer, I will present a combinatorial recipe that relates the L-function to geometric invariants of the relevant spherical variety.

3.1 Problems

We stated the "ideological thesis" that X (our spherical variety) should be affine, and we will not discuss the local problems of defining the appropriate Schwartz space $\mathcal{S}(X_v)$ and performing the unramified calculation on its "basic vector". But first, let us see an example where $X^\bullet$ not affine doesn't work:

Example 3.1.1. Let $X^\bullet$ be the quotient of $(\mathrm{SL}_2)^3$ by the subgroup H_3, where:

$$H_n = \left\{ \begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & x_n \\ & 1 \end{pmatrix} \times a \middle| x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

Let $G = (\mathbb{G}_m \times (\mathrm{SL}_2)^3)/\{\pm 1\}$, where $\mathbb{G}_m$ acts by left multiplication by the diagonal of $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, and let $H \subset G$ be the stabilizer of a point on X.

One can show that the H-period integrals converge for $\Re\pi \gg 0$ (this refers to the character of $\mathbb{G}_m$) and are Eulerian. However, when one computes the local unramified factors (i.e. when one computes $\mathcal{P}_{X^\bullet}$ applied to the basic vector $1_{X^\bullet(\mathfrak{o}_v)}$ then one doesn't get an expression which should have analytic continuation (as the automorphic character of $\mathbb{G}_m$ varies in a complex family).

On the other hand, if we take $X = \overline{X^\bullet}^{\mathrm{aff}}$, there is an exceptional isomorphism of this space with the affine closure of $[S,S]\backslash \mathrm{Sp}_6$, and as we will discuss below the Schwartz space is known in this case (equipped with a basic vector Φ_v^0 at almost every place). The local factors of the Euler product for $|\mathcal{P}_X|^2$ are equal to:

$$\frac{L(\pi_0, \times, \frac{1}{2}+s) L(\tilde{\pi}, \times, \frac{1}{2}-s)}{L(\pi_0, \mathrm{Ad}, 1)},$$

where $\times$ denotes the obvious tensor product representation of the dual group, and π is assumed to be a twist of a unitary representation π_0 by $|\chi|^s$, where χ is a suitable generator of the character group of G.

This is Garrett's triple-product construction. Similarly, the cases $n=1$ and $n=2$ give Hecke periods and Rankin-Selberg integrals (giving to the cases $n=1$, resp. $n=2$, of the corresponding L-function).

3.2 Cases where the Schwartz space is known

Very few:

1. When X is smooth affine, in which case we use the usual Schwartz space of rapidly decaying smooth functions. In this case, X is necessarily a vector bundle over a homogeneous affine variety (s. classification by Knop and Van Steirteghem).

   For example, when $X^\bullet = P_n \backslash \mathrm{GL}_n$, where P_n is the mirabolic subgroup (and $\tilde{P}_n$ the corr. parabolic) then $X = \overline{X^\bullet}^{\mathrm{aff}}$ is an n-dimensional vector space. If $\Phi \in \mathcal{S}(X(\mathbb{A}_k))$ and we integrate $\Sigma\Phi$ against an idele class character χ of the center, we get an Eisenstein series $E(\Phi,\chi)$. Compared to a "Langlands

Eisenstein series" $E(f_\chi)$, where $\chi \mapsto f_\chi \in I_{\tilde{P}_n}^G(\chi)$ is a "constant" section (constant on a fixed maximal compact subgroup),$E(\Phi,\chi)$ differs by a partial Dirichlet L-factor times a factor that depends on choices at a finite number of places.

For example, in the case $n = 2$ this corresponds to the difference between classical Eisenstein series defined by sums over the integers of the form:

$$\sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}} \text{ vs. } \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}.$$

The left hand side is the right hand side times $\zeta(2s)$.

2. Generalizing the last example, Braverman and Kazhdan have defined the full Schwartz spaces for varieties of the form $X = \overline{[P,P]\backslash G}^{\text{aff}}$, where G is split and P is a parabolic.

3. In the rest of the cases I am familiar with, only the basic vector Φ_v^0 is known, obtained by a construction that we will discuss. Notably, Braverman, Finkelberg, Gaitsgory and Mirkovic have considered the varieties:
$$X = \overline{U_P\backslash G}^{\text{aff}}$$
as $M \times G$-spherical varieties, where $P = MU_P$ is a parabolic and G is split with simply connected derived group. The basic vector can be formulated in terms of its relation to $1_{X(\mathfrak{o}_v)}$, namely there is an element in the unramified Hecke algebra of M which acting on the left (normalized-unitary action) on Φ_v^0 gives $1_{X(\mathfrak{o}_v)}$. The Satake transform of this element is the polynomial function sending a Satake parameter $c \in \check{M}$:
$$\det(1 - q^{-1}c|_{\check{\mathfrak{u}}_P}).$$
Thus, the Eisenstein series obtained from those functions after integrating against a representation τ of M will differ from Langlands' by partial L-functions of the form:
$$L(\tau, \check{\mathfrak{u}}_P, 1).$$

4. The basic function is known now for affine toric varieties, as I will describe below.

5. It is known for simple reductive monoids (also below).

3.3 Construction of the basic function

If q denotes the residual characteristic at a nonarchimedean place v, the idea is to use a scheme $\mathfrak{X}$ over $\mathbb{F}_q$ such that $\mathfrak{X}(\mathbb{F}_q) = X(\mathfrak{o}_v)$, and obtain the basic function Φ^0 by applying the function-sheaf dictionary (alternating trace of Frobenius on the stalks over $\mathbb{F}_q$-points) to the intersection cohomology sheaf $IC_{\mathfrak{X}}$.

There are difficulties, however, implementing this, as the scheme $\mathfrak{X}$ will be infinite-dimensional and there is no appropriate theory of intersection cohomology yet in this setting.

Instead, one considers a global model, but for this we will need to pass to equal characteristic. Thus, we use a Cartan decomposition of the form:

$$(X(\mathfrak{o}_v) \cap X(k_v))/G(\mathfrak{o}_v) \leftrightarrow \Lambda_X^+$$

(some monoid), which holds over both F_v and $\mathbb{F}_q((t))$ to transfer functions in a rather ad hoc way from one to the other. Hence, from now we will assume that $\mathfrak{o}_v = \mathbb{F}_q[[t]]$, and that X is defined over $\mathbb{F}_q$.

Then points of $X(\mathfrak{o}_v)$ can be thought of as maps from the formal disk $\mathrm{specf}(\mathbb{F}_q[[t]])$ to X, and we will replace the formal disk by a smooth projective curve C over $\mathbb{F}_q$. We define:

$$\mathcal{Z} := \mathrm{Maps}(C \to X/G),$$

the moduli stack parametrizing isomorphism classes of pairs $(\mathcal{P}_G, \sigma)$, where $\mathcal{P}_G$ is a principal G-bundle over C and σ is a section: $C \to \mathcal{X} \times^G \mathcal{P}_G$. We also require that σ maps generically to the open orbit $X^\bullet \times^G \mathcal{P}_G$.

For such a σ and any closed point $c \in |C|$ we have a well-defined valuation:

$$\mathrm{val}_c(\sigma) \in (X(\mathfrak{o}_v) \cap X(k_v))/G(\mathfrak{o}_v) = \Lambda_X^+$$

as follows: if we trivialize the bundle in a formal neighborhood of c, and identify this with the formal disk D, we get an element of $(X(\mathfrak{o}_v) \cap X(k_v))$; modding out by the trivialization, we get a well defined element of Λ_X^+.

Now, we can consider the intersection cohomology complex $IC_{\mathcal{Z}}$ (properly normalized) of $\mathcal{Z}$ and obtain a function $\Phi_{\mathcal{Z}}$ on $\mathcal{Z}(\mathbb{F}_q)$ (i.e. isomorphism classes of data as above defined over $\mathbb{F}_q$) by taking the alternating trace of Frobenius. The hope is that this function is factorizable, i.e. there is a function Φ^0 on Λ_X^+ (equal to 1 on the identity element) such that:

$$\Phi_{\mathcal{Z}}(\mathcal{P}_G, \sigma) = \prod_{c \in |C|} \Phi^0(\mathrm{val}_c(\sigma)).$$

We then take this function Φ^0 to be the basic function of our Schwartz spaces.

Example 3.3.1. If $X = G = \mathbb{G}_m$ then we are classifying $\mathbb{G}_m$-bundles together with a section. Such pairs of data are unique up to isomorphism (the section defines a canonical trivialization of the bundle), and hence the moduli space $\mathcal{Z}$ is a point. The resulting function is the characteristic function of $\mathfrak{o}_v^\times$.

On the other hand, if $G = \mathbb{G}_m$ and $X = \mathbb{A}^1$, then we are classifying line bundles with a section. Such a pair is completely determined by the zero divisor of the section, and hence our moduli space is the space $\mathrm{Div}(C)$ of effective divisors on C. We have:

$$\mathrm{Div}(C) = \sqcup_{m \geqslant 0} C_m,$$

where C_m denotes the m-th symmetric power of the curve (the GIT quotient $C^m /\!\!/ S_m$). The resulting function is the characteristic function of $\mathfrak{o}_v$.

In the general case, $\mathcal{Z}$ will not be a scheme but an Artin stack of locally finite type. However, if X is smooth then so is $\mathcal{Z}$, and the resulting function will be the characteristic function of $X(\mathfrak{o}_v)$. If X has singularities, the function may blow up there.

3.4 Affine toric varieties

An affine toric variety is determined by a group Λ_X of cocharacters and a strictly convex, finitely generated cone Λ_X^+ inside of it.

In this case, a calculation that we performed with Ngô B.C. showed that the basic function is:

$$\Phi^0 = \prod_i \frac{1}{1 - e^{\check{\lambda}_i}},$$

where the $\check{\lambda}_i$'s are the primitive (indecomposable) elements in Λ_X^+.

The moduli space is again a scheme, which locally looks roughly like intersections of the symmetric powers C_m that we saw above.

3.5 Simple reductive monoids

In the group case $X^{\bullet} = H$, if we let Λ_H denote the coweights of the universal Cartan of H then an affine embedding of H (automatically a monoid) is determined by a strictly convex cone spanned by the positive root cone $\mathcal{R}$ and a finite number of nonzero antidominant elements. In particular, semisimple groups have no nontrivial affine embeddings.

For the simple monoids considered by Ngô, the character group of H is generated by an element $\det$, and the cone is spanned by an element $\check{\rho} \in \Lambda_H$ with $\langle \check{\rho}, \det \rangle = 1$ (in additive notation). Let us denote the corresponding monoid by $X = H_{\check{\rho}}$.

In this setting he conjectured, and recently proved, that the basic function is well defined and the element of the Hecke with the following Satake transform takes it to $1_{H(\mathfrak{o}_v)}$:

$$\det(1 - \bullet|_{V_{\check{\rho}}}),$$

where $V_{\check{\rho}}$ is the representation of the dual group with lowest weight $\check{\rho}$. Thus, the Godement-Jacquet construction for this case gives the L-function $L(\pi_0, V_{\check{\rho}}, s)$ (as we vary an automorphic representation π_0 by powers of $|\det|$).

Of course, that reminds us of the basic and most difficult problem that we will face:

The global pairing $\mathcal{P}_X$ converges only for $\Re(\pi) \gg 0$, and we have no idea how to analytically continue it!

3.6 Unramified calculation for periods

In general, by unramified calculation we mean the computation of the basic function and that of the local periods J_{π_v} of the Plancherel formula (s. Conjectures 1, 2) applied to it. Notice that the unramified Plancherel formula (the Plancherel formula for $L^2(X_v)^{G(\mathfrak{o}_v)}$) is known, so we do not have to wait for a resolution of the full relative local Langlands conjecture in order to compute the J_{π_v}s. Set $F = k_v$.

First, let us discuss the affine homogeneous case $X = X^{\bullet}$ – then the basic function is $1_{X(\mathfrak{o}_v)}$. What are the local factors for $|\mathcal{P}_X|^2$? The answer, known in the split case at least (we will assume now that G is split), will depend on combinatorial data for X. First piece of data is the torus $\check{A}_X$ of "X-admissible" unramified characters, the maximal torus of the connnected dual group $\check{G}_X$ of X. Its cocharacter group is the group generated by highest weights of the Borel subgroup on the coordinate ring $F[X]$, and its character group will be denoted by Λ_X.

These highest weights appearing on $F[X]$, now, define a "cone" (saturated monoid in the group they generate), and the dual cone $\mathcal{R}_X$ is spanned by a set of extremal rays, which has a canonical set of generators $\{\check{v}_D\}_D \subset \Lambda_X$. (This choice of generators is dictated by the valuations induced by colors – irreducible B-stable divisors on $X^{\bullet}$; sometimes, they have to be counted with multiplicity.) The cone $\mathcal{R}_X$ is some analog of the positive root cone in the group case, and in the affine homogeneous case, the $\langle W_X, \{\pm 1\}\rangle$-closure Θ of this set has remarkable properties (not completely proven, but easy to check in each case), similar to properties of root systems. (Here W_X is the Weyl group of $\check{G}_X$, which we will not explain.) That allows one to define a virtual L-value for $\check{G}_X$ by the formula:

$$\frac{\prod_{\check{\gamma} \in \check{\Phi}_X^+} (1 - q^{-1} e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta^+} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})}.$$

The constants $r_{\check{\theta}}$ are defined when $\check{\theta} = \check{v}_D$ as:

$$\left\langle \check{v}_D, \rho_{P(X)} \right\rangle$$

and extended to all Θ by $\langle W_X, \{\pm 1\}\rangle$-invariance. The parabolic $P(X)$ is the stabilizer of the open Borel orbit, and is related to the "Arthur SL_2" that we saw above. (It corresponds to a standard Levi $\check{L} \subset \check{G}$, and SL_2 maps principally into that Levi.) Finally, the sign $\sigma_{\check{\theta}} \in \{\pm 1\}$ will not be defined here.

Up to local abelian zeta factors which do not depend on the representation, these are the local factors of $|\mathcal{P}_X|^2$.

Example 3.6.1. For $\mathrm{PGL}_2^{\mathrm{diag}} \backslash (\mathrm{PGL}_2)^3$ the dual group is $(\mathrm{SL}_2)^3$ and there are three colors inducing valuations:

$$\check{v}_1 = \frac{-\check{\alpha}_1 + \check{\alpha}_2 + \check{\alpha}_3}{2},$$

$$\check{v}_2 = \frac{\check{\alpha}_1 - \check{\alpha}_2 + \check{\alpha}_3}{2},$$

$$\check{v}_3 = \frac{\check{\alpha}_1 + \check{\alpha}_2 - \check{\alpha}_3}{2}.$$

We have $P(X) = B$ so $\left\langle \check{v}_i, \rho_{P(X)} \right\rangle = \frac{1}{2}$ for all i, and the $\langle W_X, \{\pm 1\}\rangle$-closure of this set is the set of all coweights of the form $\frac{\pm\check{\alpha}_1 \pm \check{\alpha}_2 \pm \check{\alpha}_3}{2}$.

These are all the nontrivial weights of the product representation of the dual group, and up to zeta factors the above formula gives a quotient of local L-values:

$$\frac{L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2})}{L(\pi, \mathrm{Ad}, 1)},$$

the local factor of the triple period of Harris and Kudla (low rank of orthogonal Gross-Prasad).