The local Gan-Gross-Prasad conjecture
Lecture 1

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F : p-adic field.

The local Gan-Gross-Prasad conjecture (for Bessel models) deals with some triples (the GGP triples) (G, H, ξ) where

- G is a connected reductive group $/F$;
- $H < G$ is an algebraic subgroup ;
- $\xi : H(F) \to \mathbb{C}^\times$ is a continuous character.

Roughly, the conjecture predicts the behavior of

$$m(\pi) = dim\ Hom_H(\pi, \xi) \text{ for } \pi \in Irr(G) = \text{smooth dual of } G(F)$$

in terms of the Langlands parametrization of $Irr(G)$. More precisely, the Gan-Gross-Prasad triples come from pairs $W \subset V$ of either quadratic spaces or hermitian spaces (in which case there is a quadratic extension E/F fixed). We will denote uniformly by h the underlying quadratic or hermitian form on these spaces. To get a GGP triple, you need the following two conditions to be satisfied :

- $W^\perp$ is odd-dimensional ;
- The orthogonal/unitary group of $W^\perp$ is quasiplit.

Actually, for hermitian spaces the second condition is automatic (since we are dealing with p-adic groups, any unitary group of odd rank is quasiplit). Anyway, in all cases, the two conditions taken together are equivalent to the existence of a basis

$$z_r, z_{r-1}, \dots, z_1, z_0, z_{-1}, \dots, z_{-r}$$

of $W^\perp$ such that

$$h(z_i, z_j) \neq 0 \Leftrightarrow i = -j$$

Fix such a basis and consider the following parabolic subgroup of $G(V) = SO(V)$ or $U(V)$:

$$P = Stab_{G(V)}\left(< z_r > \subsetneq < z_r, z_{r-1} > \subsetneq \ldots \subsetneq < z_r, \ldots, z_1 >\right)$$

Remark that $G(W)(= SO(W)$ or $U(W))$ is a subgroup of P. Denote by $N = Rad_u(P)$ the unipotent radical of P. The subgroups N and P have more concrete descriptions in terms of matrices : considering a basis of the form $z_r, \ldots, z_1, W \oplus < z_0 >, z_{-1}, \ldots, z_{-r}$, P is the subgroup of upper triangular by block matrices of a certain form and N the subgroup of unipotent upper triangular by block matrices of the same form. Exercise : draw the shape of these (unipotent) upper triangular by block matrices.

Fix a nontrivial additive character $\psi : E \to \mathbb{C}^\times$ (with $E = F$ in the orthogonal case) and set

$$\xi(n) = \psi\left(\sum_{i=0}^{r-1} h(nz_i, z_{-i-1})\right)$$

for all $n \in N(F)$. This defines a character $\xi : N(F) \to \mathbb{C}^\times$ which is invariant by $G(W)(F)$-conjugation and so may be extended to a character

$$\xi : N(F)G(W)(F) = N(F) \rtimes G(W)(F) \to \mathbb{C}^\times$$

by setting $\xi(ng_W) = \xi(n)$. Now, take

$$G = G(V) \times G(W)$$

$$H = N \rtimes G(W)$$

and define an embedding $H \hookrightarrow G$ by $ng_W \mapsto (ng_W, g_W)$. We got a triple (G, H, ξ) and all triples obtained this way will be called GGP triples. The GGP conjectures are relative to these triples.

As before, we define a multiplicity

$$m(\pi) := dim\ Hom_H(\pi, \xi)$$

for $\pi \in Irr(G)$. This multiplicity is easily seen to depends only on the pair of quadratic/hermitian spaces (V, W) and on π (so it doesn't depend on the particular inclusion $W \subset V$, the choice of a basis $z_r, \ldots, z_{-r}$ and the character ψ). Let me give two particular examples of the situation :

- If $W = 0$, then $G = G(V)$, N is a maximal unipotent subgroup of G and ξ is a generic character of $N(F)$. This is called the Whittaker case because then $m(\pi)$ is just the dimension of the space of Whittaker functionals for π (wrt (N, ξ)) ;
- If $dim(W) = dim(V) - 1$, then $N = 1$ and $\xi = 1$ and the embedding of $H = G(W)$ into $G = G(V) \times G(W)$ is the diagonal one. If $\pi = \pi_V \otimes \pi_W$ then we have

$$Hom_H(\pi, 1) = Hom_{G(W)}(\pi_V, \pi_W^\vee)$$

Hence we are studying the branching-law problem of knowing which irreducible representations of $G(W)(F)$ appear as quotient of $\pi_{V|G(W)}$. This will be called the <u>codimension one case</u>.

Back to the general case, here is a basic foundational result of the subject

Theorem 1 (Aizenbud-Gourevitch-Rallis-Schiffmann, Waldspurger) *We always have*

$$m(\pi) \leqslant 1$$

Remark that in the Whittaker case this is equivalent to the unicity of the Whittaker model.

As I said, roughly the GGP conjectures predicts for which π we get the multiplicity to be one. These conjecture are build on the Local Langlands Correspondence (LLC) seen as a way to parametrize $Irr(G)$.

<u>**LLC (rough version)**</u> : There should exist a decomposition

$$Irr(G) = \bigsqcup_{\varphi} \Pi^G(\varphi)$$

into finite sets called L-packets. The decomposition begin indexed by some set of morphisms $\varphi : WD_F \to {}^L G$ called Langlands parameters. Moreover, to each Langlands parameter is associated a finite 2-abelian group A_φ and it is conjectured that there should exist bijections

$$\Pi^G(\varphi) \simeq \widehat{A}_\varphi^G \subseteq \widehat{A}_\varphi$$
$$\pi(\varphi, \chi) \hookleftarrow \chi$$

where $\widehat{A}_\varphi$ is the group of characters on A_φ and $\widehat{A}_\varphi^G$ is a certain subset of it. Of course, this parametrization of $Irr(G)$ have to satisfy some expected properties.

I will come back to this and state precisely what we need from LLC later. Now assuming LLC, we can state the local GGP conjecture informally as follows

<u>**GGP conjecture (rough version)**</u> : For all generic Langlands parameter φ, there exists at most one $\pi \in \Pi^G(\varphi)$ such that $m(\pi) = 1$ and moreover, if such a π exists, we can describe the corresponding character χ in terms of certain ϵ factors.

We will also be more precise about the conjecture later in the course.

What has been done :
- In a serie of four papers, Waldspurger proved the conjecture for tempered L-packets in the orthogonal case;
- In an additional paper, Moeglin and Waldspurger reduced the generic case to the tempered one (so that the orthogonal case is completely done);

– In my thesis, I adaptated the method of Waldspurger to deal with tempered representations in the unitary case

Since the proofs are very similar in both cases, I will from now on focus on the unitary case. I will also restrict myself to tempered representations. So let (G, H, ξ) be an unitary GGP triple and denote by

$$Temp(G) \subset Irr(G)$$

the tempered dual of $G(F)$. The proof of the conjecture roughly goes like that

– We prove integral formulas for the multiplicity $m(\pi)$, $\pi \in Temp(G)$, as well as for certain ϵ factors ;
– Using LLC and its endoscopic characterization, we are able to relate the two previous formulas and so find an expression of the multiplicity $m(\pi)$ in terms of certain ϵ factors. This relation happens miraculously to be exactly the one predicted by GGP !

1 A formula for $m(\pi)$

1.1 The Theorem

The formula I want to discuss express $m(\pi)$, for $\pi \in Temp(G)$, in term of the Harish-Chandra character θ_π of π. So I will start by recalling some facts about this character (beginning with its definition).

Let $\pi \in Irr(G)$. For $f \in C_c^\infty(G(F))$, we may define the operator

$$\pi(f) = \int_{G(F)} f(g)\pi(g)dg$$

This operator is of finite rank (since π is automatically admissible) and this allows us to define a distribution θ_π on $G(F)$ by setting

$$\theta_\pi(f) = trace\, \pi(f), \quad \forall f \in C_c^\infty(G(F))$$

It is a deep theorem of Harish-Chandra that θ_π is in fact represented by a locally integrable function $\theta_\pi \in L^1_{loc}(G(F))$:

$$\theta_\pi(f) = \int_{G(F)} f(g)\theta_\pi(g)dg$$

It is this function that is called the Harish-Chandra character of π. The character is always locally constant (smooth) on the regular locus $G_{reg}(F)$. Harish-Chandra went further and described rather explicitly the local behavior of θ_π near every semisimple point $x \in G_{ss}(F)$. This can be stated as follow

Theorem 2 *For all $x \in G_{ss}(F)$, we have a local expansion*

$$\theta_\pi(xe^X) = \sum_{\mathcal{O}\in Nil(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x)\hat{j}(\mathcal{O},X)$$

for almost all $X \in \omega$. Where :
- *$G_x = Z_G(x)$ and $\mathfrak{g}_x = Lie(G_x)$;*
- *$\omega \subset \mathfrak{g}_x(F)$ is an open neighborhood of 0 ;*
- *$Nil(\mathfrak{g}_x)$ denotes the set of all $G_x(F)$-nilpotent orbits in $\mathfrak{g}_x(F)$;*
- *For $\mathcal{O} \in Nil(\mathfrak{g}_x)$, $\hat{j}(\mathcal{O},.)$ is the Fourier transform of the orbital over $\mathcal{O}$ (see below for more explanations) ;*
- *$c_{\pi,\mathcal{O}}(x)$ are just complex numbers.*

Let me explain a little bit further what is the function $\hat{j}(\mathcal{O},.)$. First, fix a $G_x(F)$-invariant symmetric and nondegenerate bicharacter $< .,. >$: $\mathfrak{g}_x(F) \times \mathfrak{g}_x(F) \to \mathbb{C}^\times$. This allows us to define a Fourier transform

$$C_c^\infty(\mathfrak{g}_x(F)) \to C_c^\infty(\mathfrak{g}_x(F))$$
$$f \mapsto \hat{f}(X) = \int_{\mathfrak{g}_x(F)} f(Y) < X,Y > dY$$

where dY is the autodual Haar measure, that is the only one such that $\hat{\hat{f}}(X) = f(-X)$.

For $\mathcal{O} \in Nil(\mathfrak{g}_x)$, we may form the integral orbital over $\mathcal{O}$ (a distribution) :

$$f \in C_c^\infty(\mathfrak{g}_x(F)) \mapsto J_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(N)dN$$

Now, as an analog for the Lie algebra of the result about characters on the group, Harish-Chandra has proved that the distribution

$$f \mapsto J_{\mathcal{O}}(\hat{f})$$

is represented by a locally integrable function $X \mapsto \hat{j}(\mathcal{O},X)$ which is locally constant on the regular locus $\mathfrak{g}_{x.reg}(F)$.

For $\pi \in Irr(G)$, define a function

$$c_\pi : G_{ss}(F) \to \mathbb{C}$$
$$c_\pi(x) = \frac{1}{\mid Nil(\mathfrak{g}_x)_{reg} \mid} \sum_{\mathcal{O}\in Nil(\mathfrak{g}_x} c_{\pi,\mathcal{O}}(x)$$

where $Nil(\mathfrak{g}_x)_{reg} \subset Nil(\mathfrak{g}_x)$ is the subset of regular nilpotent orbits. There are no regular nilpotent orbit if $\mathfrak{g}_x$ is not quasisplit and so in this case we have $c_\pi(x) = 0$. Note that for $x \in G_{reg}(F)$, we have

$$c_\pi(x) = \theta_\pi(x)$$

For $\pi \in Temp(G)$, the formula will express $m(\pi)$ by means of the function c_π. Before stating the result, it remains to define a space (of conjugacy classes) over which we are going to integrate c_π.

Let $\mathcal{C}(V,W)$ be the following set of conjugacy classes in $H_{ss}(F) = U(W)_{ss}(F)$:

$$\mathcal{C}(V,W) = \left(\bigsqcup_{W' \subseteq W \ nondeg} U(W')(F)_{ell} \right) / U(W)(F)$$

where the union is over the set of nondegenerate subspaces $W' \subset W$. Here $U(W')(F)_{ell}$ denote the elliptic regular locus i.e. the subset of elements $x \in U(W')_{reg}(F)$ such that $U(W')_x(F)$ is compact. Note that since $U(W)(F)$-conjugation in $U(W')(F)$ is the same than $U(W')(F)$-conjugation, we may rewrite

$$\mathcal{C}(V,W) = \bigsqcup_{W' \ nondeg/U(W)(F)} U(W')(F)_{ell}/conj$$

where this time the disjoint union is over the set of $U(W)(F)$-orbits of nondegenerate subspaces $W' \subset W$. Fix on $U(W')(F)_{ell}/conj$ the measure defined by

$$\int_{U(W')(F)_{ell}/conj} \varphi(x)dx = \sum_{T \in \mathcal{T}_{ell}(U(W'))} \mid W(T) \mid^{-1} \int_{T(F)} \varphi(x)dx$$

where $\mathcal{T}_{ell}(U(W'))$ is a set of representatives for the conjugacy classes of maximal elliptic tori in $U(W')$, $W(T)$ is the Weyl group of T and the measure on $T(F)$ is the unique Haar measure with total mass 1. Equip $\mathcal{C}(V,W)$ with the disjoint union of these measures. Now define

$$m_{geom}(\pi) := \lim_{s \to 0^+} \int_{\mathcal{C}(V,W)} D^H(x) c_\pi(x) \Delta(x)^s dx$$

where D^H is the absolute value of the usual Weyl discriminant :

$$D^H(x) = \mid det(1 - Ad(x))_{|\mathfrak{h}/\mathfrak{h}_x} \mid$$

and Δ is defined by

$$\Delta(x) = \mid det(x-1)_{|W/W(x)} \mid$$

$(W(x) = Ker(x-1))$.

Theorem 3 *(i) This expression makes sense : the integral is absolutely convergent for $Re(s) > 0$ and the limit as $s \to 0$ exists ;*

(ii) If $\pi \in Temp(G)$, we have an equality

$$m(\pi) = m_{geom}(\pi)$$

This is the announced formula for the multiplicity. Before going into its proof, let me give first an application to (a crude version of) the GGP conjecture.

1.2 First application to GGP

To state it I need of course a first version of LLC. In fact, the GGP conjecture is best expressed in a form of LLC enhanced by Vogan where we consider a group together with all its pure inner forms at the same time. The pure inner forms of a unitary group $U(V)$ are the unitary groups $U(V')$ where V' is an hermitian space of the same dimension than V.

Assume the group G we were working with is quasisplit and affect with an i in index all the objects previously introduced : $(G_i, H_i, \xi_i, V_i, W_i)$. I assume moreover, to simplify the discussion, that $W_i \neq 0$. Then there exists (up to isomorphism) exactly one hermitian space W_a with the following properties :

– $dim(W_a) = dim(W_i)$;

– $W_a \not\simeq W_i$.

Of course, there is also an unique hermitian space V_a of the same dimension than V_i but not isomorphic to it. Moreover, the hermitian space W_a may be embedded in V_a :

$$W_a \hookrightarrow V_a$$

So that we get as before a GGP triple (G_a, H_a, ξ_a) well defined up to conjugation by $G_a(F)$.

The GGP conjecture is now easier to state by considering the two triples (G_i, H_i, ξ_i) and (G_a, H_a, ξ_a) at the same time.

<u>**LLC (second version)**</u> : G_i and G_a share the same L-group ${}^LG$. There should exist two decompositions

$$Temp(G_i) = \bigsqcup_{\varphi} \Pi^{G_i}(\varphi)$$

$$Temp(G_a) = \bigsqcup_{\varphi} \Pi^{G_a}(\varphi)$$

Both indexed by the subset of the Langlands parameters $\varphi : WD_F \to {}^LG$ that are tempered. Moreover these decompositions should satisfy the following properties :

(1) For all tempered Langlands parameter $\varphi : WD_F \to {}^LG$, the L-packets $\Pi^{G_i}(\varphi)$ and $\Pi^{G_a}(\varphi)$ are finite sets ;

(2) The characters

$$\theta_{\varphi}^{G_i} = \sum_{\pi \in \Pi^{G_i}(\varphi)} \theta_\pi$$

$$\theta_{\varphi}^{G_a} = \sum_{\pi \in \Pi^{G_a}(\varphi)} \theta_\pi$$

are stable;

(3) The transfer of $\theta_\varphi^{G_i}$ to $G_a(F)$ is $-\theta_\varphi^{G_a}$;

(4) For every Whittaker datum, $\Pi^{G_i}(\varphi)$ contains exactly one representation having a Whittaker model with respect to this Whittaker datum.

Before going further, let me discuss the meaning of the last three conditions.

Two elements of $G_{\natural,reg}(F)$ ($\natural = i$ or a) are said to be stably conjugate if they are conjugate in $G_\natural(\overline{F})$. Now we say of a function on $G_{\natural,reg}(F)$ that it is stable if this function is constant on the stable conjugacy classes. This explain condition (2).

Denote by $G_{\natural,reg}(F)/stab$ the set of stable conjugacy classes in $G_{\natural,reg}(F)$. To explicit condition (3), we first need to define an injection

$$G_{a,reg}(F)/stab \hookrightarrow G_{i,reg}(F)/stab$$

Fix an algebraic closure $\overline{F}/F$ and set $\overline{E} = E \otimes_F \overline{F}$. Then the two $\overline{E}$-hermitian spaces $V_{i,\overline{E}}$ and $V_{a,\overline{E}}$ are isomorphic. Fixing such an isomorphism, we get an isomophism of groups

$$U(V_i)_{\overline{F}} = U(V_{i,\overline{E}}) \simeq U(V_{a,\overline{E}}) = U(V_a)_{\overline{F}}$$

well defined up to conjugacy, I claim that this bijection sends $U(V_a)_{reg}(F)/stab$ into $U(V_i)_{reg}(F)/stab$. Hence, we get an injection

$$U(V_a)_{reg}(F)/stab \hookrightarrow U(V_i)_{reg}(F)/stab$$

Fact (for later use) : When further restricted to $U(V_a)(F)_{ell}/stab$, this injection becomes a bijection

$$U(V_a)(F)_{ell}/stab \simeq U(V_i)(F)_{ell}/stab$$

Similarly, we have an injection

$$U(W_a)_{reg}(F)/stab \hookrightarrow U(W_i)_{reg}(F)/stab$$

(which of course also gives a bijection between the elliptic stable conjugacy classes). Taking the product of these two injections, we obtain an embedding

$$G_{a,reg}(F)/stab \hookrightarrow G_{i,reg}(F)/stab$$

which is the one we were looking for. Now, we may restate condition (3) more concretely as follows :

(3) For all $x_a \in G_{a.reg}(F)/stab$ with image $x_i \in G_{i,reg}(F)/stab$, we have

$$\theta_\varphi^{G_i}(x_i) = -\theta_\varphi^{G_a}(x_a)$$

Last but not least, let me explain condition (4). A Whittaker datum for G_i is a conjugacy class of pairs (U_B, ξ_B) where U_B is the unipotent radical of some Borel subgroup B and $\xi_B : U_B(F) \to \mathbb{C}^\times$ is a generic character on it (generic means that the stabilizer has minimal dimension). Whittaker data for G_i exist because G_i is quasisplit. We say that $\pi \in Irr(G_i)$ admits a Whittaker model with respect to a Whittaker datum (U_B, ξ_B) if

$$Hom_{U_B}(\pi, \xi_B) \neq 0$$

(by the theory of Whittaker models, this space is always of dimension at most 1). Actually, there is a bijection between the set of Whittaker data and the set of regular nilpotent orbits in $\mathfrak{g}_i(F)$. This bijection is given as follows. Fix a $G_i(F)$-invariant nondegenerate symmetric bicharacter $< .,. >$: $\mathfrak{g}_i(F) \times \mathfrak{g}_i(F) \to \mathbb{C}^\times$ (the same that we used to define the coefficients $c_{\pi,\mathcal{O}}(1)$). Let $\mathcal{O} \in Nil(\mathfrak{g}_i)_{reg}$ and pick $\overline{N} \in \mathcal{O}$. You may complete $\overline{N}$ into an $\mathfrak{sl}_2$ triple $(\overline{N}, H, N)$ where N is also regular nilpotent. Since both $\overline{N}$ and N are regular nilpotent they belong to unique Borel subalgebras $\overline{\mathfrak{b}}$ and $\mathfrak{b}$ respectively. Let U_B be the unipotent radical of the Borel subgroup with Lie algebra $\mathfrak{b}$ and define a character ξ_B of $U_B(F)$ by

$$\xi_B(e^N) =< \overline{N}, N >$$

for all $N \in \mathfrak{u}_B$. This is a generic character and the map $\mathcal{O} \mapsto (U_B, \xi_B)$ yields the desired bijection. Using this parametrization, Rodier gave the following formula for $dim\ Hom_{U_B}(\pi, \xi_B)$.

Theorem 4 (Rodier) *For all $\mathcal{O} \in Nil_{reg}(\mathfrak{g}_i)$ with corresponding Whittaker datum (U_B, ξ_B), we have*

$$c_{\pi,\mathcal{O}}(1) = dim\ Hom_{U_B}(\pi, \xi_B)$$

By this formula of Rodier, we see that condition (4) implies the following condition (4') :

$$(4') \qquad \sum_{\pi \in \Pi^{G_i}(\varphi)} c_\pi(1) = 1$$

We are now in position to prove and state a first version of GGP :

Theorem 5 (GGP (second version)) *Let $\varphi : WD_F \to {}^LG$ be a tempered Langlands parameter and assume the existence of two finite sets $\Pi^{G_i}(\varphi) \subset Temp(G_i)$ and $\Pi^{G_a}(\varphi) \subset Temp(G_a)$ satisfying the conditions (1)-(4) above. Then there exists a unique $\pi \in \Pi^{G_i}(\varphi) \sqcup \Pi^{G_a}(\varphi)$ such that*

$$m(\pi) = 1$$

<u>Proof</u> : Set $\Pi_\natural = \sum_{\pi \in \Pi^{G_\natural}(\varphi)} \pi$ ($\natural = i$ or a). Extend $\pi \mapsto m(\pi)$ by linearity to all virtual representations. Then we want to prove that

$$m(\Pi_i) + m(\Pi_a) = 1$$

Define the functions $\theta_\natural = \sum_{\pi \in \Pi^{G_\natural}(\varphi)} \theta_\pi$ and $c_\natural = \sum_{\pi \in \Pi^{G_\natural}(\varphi)} c_\pi$. Then the formula for the multiplicity express $m(\Pi_\natural)$ in terms of the function $c_\natural$. More precisely, we have

$$m(\Pi_\natural) = \lim_{s \to 0^+} \int_{\mathcal{C}(V_\natural, W_\natural)} D^{H_\natural}(x) c_\natural(x) \Delta(x)^s dx$$

Since $\Theta_\natural$ is stable (condition (2)), we can rearrange the previous expression slightly differently by invoking stable conjugacy classes rather than just conjugacy classes. Recall that

$$\mathcal{C}(V_\natural, W_\natural) = \bigsqcup_{W' \subset W_\natural \text{ nd}/U(W_\natural)(F)} U(W')(F)_{ell}/conj$$

Let's define

$$\mathcal{C}_{stab}(V_\natural, W_\natural) = \bigsqcup_{W' \subset W_\natural \text{ nd}/U(W_\natural)(F)} U(W')(F)_{ell}/stab$$

We have an obvious projection

$$\mathcal{C}(V_\natural, W_\natural) \to \mathcal{C}_{stab}(V_\natural, W_\natural)$$

with finite fibers. Moreover, the function $c_\natural$ is obviously constant on the fibers so we may rewrite

$$m(\Pi_\natural) = \lim_{s \to 0^+} \int_{\mathcal{C}_{stab}(V_\natural, W_\natural)} D^{H_\natural}(x) c_\natural(x) \Delta(x)^s dx$$

We will now compare the two expressions (for $\natural = i$ or a). First set

$$\mathcal{C}_{stab}(V_\natural, W_\natural)^* := \bigsqcup_{W' \neq 0} U(W')(F)_{ell}/stab$$

Then I claim that we have a natural bijection

$$\mathcal{C}_{stab}(V_a, W_a)^* \simeq \mathcal{C}_{stab}(V_i, W_i)^*$$

extending the one already defined

$$U(W_a)(F)_{ell}/stab \simeq U(W_i)(F)_{ell}/stab$$

We say that two nondegenerate subspaces $W'_a \subseteq W_a$ and $W'_i \subseteq W_i$ correspond to each other if they have same dimension but aren't isomorphic as hermitian spaces. Every $U(W_a)(F)$-orbit of nonzero nondegenerate subspaces $W'_a \subseteq W_a$ corresponds to exactly one $U(W_i)(F)$-orbit of nonzero nondegenerate subspaces $W'_i \subseteq W_i$ and conversely. This yields a bijection

$$\{0 \neq W' \subseteq W_a \text{ nondeg}\}/U(W_a)(F) \simeq \{0 \neq W' \subseteq W_i \text{ nondeg}\}/U(W_i)(F)$$

$$W'_a \mapsto W'_i$$

For W'_a and W'_i corresponding, the situation is the same than for W_a and W_i (or for V_a and V_i), so that we get as before a natural bijection between elliptic stable conjugacy classes :

$$U(W'_a)(F)_{ell}/stab \simeq U(W'_i)(F)_{ell}/stab$$

Taken all together, these bijections give a one-to-one correspondence

$$\mathcal{C}_{stab}(V_a, W_a)^* \simeq \mathcal{C}_{stab}(V_i, W_i)^*$$

$$x_a \mapsto x_i$$

Fact(not hard) : Condition (3) extends naturally to give the relations

$$c_i(x_i) = -c_a(x_a)$$

for all $x_a \in \mathcal{C}_{stab}(V_a, W_a)^* \mapsto x_i \in \mathcal{C}_{stab}(V_i, W_i)^*$.

We also have $D^{H_i}(x_i) = D^{H_a}(x_a)$ and $\Delta(x_i) = \Delta(x_a)$ so that the contributions of x_i and x_a in respectively $m(\Pi_i)$ and $m(\Pi_a)$ cancel each other in the sum $m(\Pi_i) + m(\Pi_a)$. Hence, after removing the cancelling terms in the sum, only the terms corresponding to $x = 1$ remain :

$$m(\Pi_i) + m(\Pi_a) = c_i(1) + c_a(1)$$

Condition (4') tells us exactly that the first term above is 1 whereas since G_a is not quasisplit, $\mathfrak{g}_a(F)$ has no regular nilpotent orbits and so the second term is zero. ■