The local Gan Gross Prasad conjecture lecture 2-3

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1.3 An approach by local trace formula

Let's consider a general triple (G, H, ξ) where G is connected reductive over F, H is an algebraic subgroup of G and $\xi : H(F) \to \mathbb{C}^\times$ is a character. Say that we want to study the multiplicity

$$m(\pi) = dim\ Hom_H(\pi, \xi), \quad \pi \in Irr(G)$$

(like in the GGP conjecture). By Frobenius reciprocity, we have

$$Hom_H(\pi, \xi) = Hom_G\left(\pi, C^\infty(H(F)\backslash G(F), \xi)\right)$$

where $C^\infty(H(F)\backslash G(F), \xi) := \{\varphi : G(F) \to \mathbb{C}; \varphi$ is smooth and $\varphi(hg) = \xi(h)\varphi(g)$ for all $h \in H(F),\ g \in G(F)\}$. The action of $G(F)$ on this space is by right translation, we will denote it by R. So $m(\pi)$ is the number of time π appears discretely in $C^\infty(H(F)\backslash G(F), \xi)$. Consequently it is natural to study this (big) representation in more details. A natural way to do this is to let act on it functions $f \in C_c^\infty(G(F))$ by convolution, i.e. set

$$R(f) = \int_{G(F)} f(g)R(g)dg$$

then $R(f)$ is a Kernel operator, i.e. its action on a function $\phi \in C_c^\infty(H(F)\backslash G(F), \xi)$, is given by

$$\left(R(f)\phi\right)(x) = \int_{H(F)\backslash G(F)} K_f(x, y)\phi(y)dy$$

where K_f, the Kernel, is given by

$$K_f(x, y) = \int_{H(F)} f(x^{-1}hy)\xi(h)dh$$

A natural way to study $C^\infty(H(F)\backslash G(F), \xi)$ is to try to compute the trace of the operator $R(f)$ (for example because then we will know the character θ_R of R and so how R decomposes).

Assume for one moment that $G(F)$ is compact, then all of this makes perfect sense. First $C^\infty(H(F)\backslash G(F),\xi)$ decomposes discretely

$$C^\infty(H(F)\backslash G(F),\xi) = \bigoplus_{\pi\in Irr(G)} m(\pi)\pi$$

and the multiplicity are (always) finite. Hence, it's not hard to deduce that $R(f)$ is of trace class and that

$$trace\ R(f) = \sum_{\pi\in Irr(G)} m(\pi) trace\ \pi(f) \quad \text{(Spec)}$$

If we are able to compute $trace\ R(f)$ in another way, then by plugging $f = f_{\pi^\vee}$ (a coefficient of $\pi^\vee$) in (Spec) we would get a formula for $m(\pi)$.

Continuing with the framework where $G(F)$ is compact, since in this case $R(f)$ is trace class and is a Kernel operator, we have

$$trace\ R(f) = \int_{H(F)\backslash G(F)} K_f(x,x)dx$$

Using the above explicit expression for the Kernel, it's not hard to transform this last formula into

$$trace\ R(f) = \int_{H(F)} \Phi(h,f)\xi(h)dh \quad \text{(Geom)}$$

where $\Phi(h,f) = \int_{G(F)} f(x^{-1}hx)dx$ is the orbital integral (I am completely ignoring the issue of normalizing measures here). By equaling the two expansions (Spec) and (Geom) of the trace, we obtain the identity

$$\sum_{\pi\in Irr(G)} m(\pi) trace\ \pi(f) = \int_{H(F)} \Phi(h,f)\xi(h)dh$$

We may now apply our program : plug $f = f_{\pi^\vee}$ in this identity. Using the classical formula $\Phi(g,f) = trace\ \pi(f)\overline{\theta_\pi(g)}$, for all $g\in G(F)$, we get

$$m(\pi) = \int_{H(F)} \overline{\theta_\pi(h)}\xi(h)dh$$

A formula that we could also have obtained by using the orthonormality properties of characters.

Back to the general case : There is one major issue if one want to do something similar in the noncompact case : in general $C^\infty(H(F)\backslash G(F),\xi)$ doesn't decompose discretely and moreover $R(f)$ is usually not of trace class. Consequently, it seems that we are no longer able to obtain an identity between a spectral side (Spec) and a geometric side (Geom) in general. There is however a way to get around this that Arthur used in its proof of a local trace formula : we shall multiply the kernel $K_f(x,x)$ by the characteristic function κ_N of some large compact subset Ω_N of $H(F)\backslash G(F)$. This allows us at least to define a distribution (a truncated trace in some sense)

$$J_N(f) = \int_{H(F)\backslash G(F)} \kappa_N(x) K_f(x,x) dx$$

in which the truncation parameter N is meant to be a positive integer. The idea is then trying to obtain two expansions of this "truncated trace" at the limit when $N \to \infty$:

$$(Spec) = \lim_{N\to\infty} J_N(f) = (Geom) \quad ?$$

As before we expect the spectral side to involve characters of representations and the geometric side to involve orbital integrals. Of course, the sequence of compact subsets $(\Omega_N)_{N\geqslant 1}$ has to be chosen carefully if we would like to obtain something interesting. One of the main reasonable assumptions we would like to make about it is :

$(\Omega_N)_{N\geqslant 1}$ is an increasing and exhaustive sequence of compact-open subsets of $H(F)\backslash G(F)$.

Actually, in the cases we will be considering we won't need much more, the hypothesis we need about (Ω_N) are very loose and so I won't discuss them. There is another point that is worst : whatever the sequence (Ω_N) is, the sequence $(J_N(f))_{N\geqslant 1}$ will almost never converge for all f. We need to restrict ourself to some particular functions f. The assumption we will most often ask about f is that f is strongly cuspidal.

Definition 1 *A function $f \in C_c^\infty(G(F))$ is strongly cuspidal if for all proper parabolic subgroup $P = MN \subsetneq G$, we have*

$$\int_{N(F)} f(mn) dn = 0$$

for all $m \in M(F)$.

Although we are mainly interested in the GGP case, I will first disgress and discuss Arthur's local trace formula. These for three reasons : first of all, Arthur's trace formula will be needed later, secondly there is also a local trace formula related to the GGP conjecture and its proof follows closely on many points Arthur's proof and finally it is also the opportunity to introduce many objects that will appear also in the GGP trace formula.

1.4 Arthur's local trace formula

Arthur's trace formula deals with Arthur's triples that are of the form

$$(G = H \times H, H, \xi = 1)$$

where the embedding $H \hookrightarrow G$ is the diagonal one. By linearity we may assume $f = \varphi \otimes \varphi' \in C_c^\infty(G(F)) = C_c^\infty(H(F)) \otimes C_c^\infty(H(F))$. Moreover, we have a natural identification $H(F)\backslash G(F) = H(F)$ and the action of $G(F) = H(F) \times H(F)$ on it is by left and right translation. The truncated trace should be

$$J_N(\varphi, \varphi') = \int_{H(F)} K_{\varphi,\varphi'}(x, x)dx$$

where

$$K_{\varphi,\varphi'}(x, y) = \int_{H(F)} \varphi(x^{-1}hy)\varphi'(h)dh$$

Since $K_{\varphi,\varphi'}(x, x)$ is invariant by translation by $Z(H)(F)$, Arthur integrates over $A_H(F)\backslash H(F)$ rather than $H(F)$ (A_H is the split part of the center of H). This is not a major issue and we shall assume that the center of $H(F)$ is compact so that the previous expression is the correct one. Arthur then proves the following statement :

Theorem 1 (Arthur) *If the sequence $(\Omega_N)_{N\geqslant 1}$ is suitably chosen (here the choice really matter), then there exists a polynomial $p(\varphi, \varphi', T) \in \mathbb{C}[T]$ such that*

$$\lim_{N\to\infty} \mid J_N(\varphi, \varphi') - p(\varphi, \varphi', N) \mid = 0$$

Arthur then proceed in giving two expansions of the constant term $p(0, \varphi, \varphi')$. One is geometric and involves orbital integral and their generalization called weighted orbital integrals, the other one is spectral and involves characters of representations and their generalization called weighted characters. The equality between these two expansions is Arthur's (noninvariant) local trace formula. However, there is one case where $p(\varphi, \varphi', T)$ is already a constant (and so $J_N(\varphi, \varphi')$ has a limit) : it is when φ is strongly cuspidal. I will now discuss the geometric and spectral expansions of Arthur in this special case.

1.4.1 Geometric expansion

This one involves orbital integrals and weighted orbital integrals. Recall the definition of the usual orbital integrals. For $x \in H_{reg}(F)$, $\varphi \in C_c^\infty(H(F))$ and the choice of a Haar measure on $H_x(F)$ ($H_x = Z_H(x)$), the orbital integral of φ at x is

$$\Phi(x, \varphi) = \Phi_H(x, \varphi) = \int_{H_x(F)\backslash H(F)} \varphi(h^{-1}xh)dh$$

A weighted orbital integral depends on the following data :
- A Levi $M \subset H$;
- A point $x \in M(F) \cap H_{reg}(F)$;
- A choice of a special maximal compact subgroup K of $H(F)$

The important property we use about K is that $H(F) = P(F)K$ for all parabolic subgroup P of H. These data being fixed, we may define a weighted orbital integral

$$\Phi_M(x,\varphi) = \int_{H_x(F)\backslash H(F)} \varphi(h^{-1}xh)v_M(h)dh$$

for all $\varphi \in C_c^\infty(H(F))$. The function v_M above is the weight and has been defined by Arthur (but we won't need its precise definition).

Now, define a measure on $H_{reg}(F)/conj$ by

$$\int_{H_{reg}(F)/conj} f(x)dx = \sum_{T\in\mathcal{T}(H)} \mid W(T) \mid^{-1} \int_{T(F)} f(t)dt$$

where $\mathcal{T}(H)$ is a set of representatives of the $H(F)$-conjugacy classes of maximal tori in H, $W(T)$ is the Weyl group and $D^H(t)$ is the absolute value of the usual Weyl discriminant. To be complete, we also need to define the measure on the tori $T(F)$ but these ones have also been used implicitly in the definition of the weighted orbital integrals. We may now state the geometric expansion of Arthur's local trace formula :

Theorem 2 (Arthur) *Let $\varphi, \varphi' \in C_c^\infty(H(F))$ and assume that φ is strongly cuspidal. Then, we have*

$$\lim_{N\to\infty} J_N(\varphi,\varphi') = \int_{H_{reg}(F)/conj} (-1)^{a_{M(x)}} D^H(x)\Phi_{M(x)}(x,\varphi)\Phi_H(x,\varphi')dx$$

where for $x \in H_{reg}(F)$, $M(x) = Z_H(A_{H_x})$ i.e. the smallest Levi containing x and $a_{M(x)} = dim(A_{M(x)})$.

<u>**Remark**</u> : Since φ is strongly cuspidal, we can prove that the weighted orbital integrals of φ doesn't depend on the choice of K. By "transport de structure" this implies that the function

$$x \in H_{reg}(F) \mapsto (-1)^{a_{M(x)}}\Phi_{M(x)}(x,\varphi)$$

is invariant by $H(F)$-conjugation. In particular, the RHS of the theorem makes sense.

Let us write the result a little bit differently. Set $\theta_\varphi(x) = (-1)^{a_{M(x)}}\Phi_{M(x)}(x,\varphi)$, $x \in H_{reg}(F)$. Then we may rewrite the result as

$$(Geom) \quad \lim_{N\to\infty} J_N(\varphi,\varphi') = \int_{H_{reg}(F)/conj} D^H(x)\theta_\varphi(x)\Phi_H(x,\varphi')dx$$

1.4.2 Spectral expansion

The spectral side is built on another type of distributions : characters and weighted characters. Recall that the usual character of $\pi \in Irr(H)$ is the distribution

$$\varphi \in C_c^\infty(H(F)) \mapsto trace\ \pi(\varphi) = \Phi_H(\pi, \varphi)$$

A weighted character depends on the following data :

- A Levi $M \subset H$;
- A representation $\sigma \in Temp(M)$;
- A special maximal compact subgroup K of $H(F)$.

As before, the only property we need about K is that $H(F) = P(F)K$ for all parabolic subgroup P of H. From these datas, Arthur construct a weighted character

$$\varphi \in C_c^\infty(H(F)) \mapsto trace\ R_M(\sigma)\pi(\varphi) = \Phi_M(\pi, \varphi)$$

where $\pi = i_P^H(\sigma)$ is the normalized parabolically induced representation from σ, $P = MU_P$ being any parabolic with Levi component M, and $R_M(\sigma)$, the "weight", is a certain operator (depending on K) constructed by Arthur

$$R_M(\sigma) : i_P^H(\sigma) \to i_P^H(\sigma)$$

Before stating the spectral side of Arthur's local trace formula, I need to introduce the spectral analog of the space of (regular) conjugacy classes. For this, I need to recall some facts about tempered representations. Denote by $Rep_{temp}(H)$ the category of finite length tempered representations of $H(F)$. Let $P = MN$ be a parabolic subgroup of H. The normalized induction is a functor

$$Rep_{temp}(M) \to Rep_{temp}(H)$$
$$\sigma \mapsto i_P^H(\sigma)$$

It turns out that this functor admits a left adjoint, the weak Jacquet module :

$$Rep_{temp}(H) \to Rep_{temp}(M)$$
$$\pi \mapsto \pi_P^w$$

So we have a functorial isomorphism

$$Hom_H(\pi, i_P^H(\sigma)) = Hom_M(\pi_P^w, \sigma)$$

for all $\pi \in Rep_{temp}(H)$, $\sigma \in Rep_{temp}(M)$. The weak Jacquet module π_P^w is cut out in the usual Jacquet module π_P by keeping only the generalized eigenspaces in π_P corresponding to unitary characters of $Z(M)(F)$ (the center of $M(F)$).

Let $R_{temp}(H)$ be the Grothendieck group of $Rep_{temp}(H)$, i.e. the vectore space over $\mathbb{C}$ with basis $Temp(G)$. Since $\pi \mapsto \pi_P^w$ is exact, it defines a morphism

$$R_{temp}(H) \to R_{temp}(M)$$
$$\pi \mapsto \pi_P^w$$

Define $R_{ell}(H) = \{\pi \in R_{temp}(H); \quad \pi_P^w = 0 \text{ for all proper parabolic } P = MN \subsetneq H\}$. Since the functor $\sigma \mapsto i_P^H(\sigma)$ is also exact, we also get morphisms

$$R_{temp}(M) \to R_{temp}(H)$$
$$\sigma \mapsto i_P^H(\sigma)$$

for all parabolic subgroup $P = MN \subseteq H$. Define $R_{ind}(H) = \bigoplus_{P=MN\subsetneq H} i_P^H\left(R_{temp}(M)\right)$. Then we have

$$R_{temp}(H) = R_{ell}(H) \oplus R_{ind}(H)$$

Moreover, Arthur has define a basis $T_{ell}(H) \subset R_{ell}(H)$ (the so called elliptic representations although they are not in general genuine representations but rather just virtual representations). Let us just say that $T_{ell}(H)$ contains the set $\Pi_2(H)$ of square-integrable irreducible representations, that this set is invariant by any automorphism θ of H as well as by twists by unitary unramified characters. This last condition allows us to equip $T_{ell}(M)$ with a natural structure of real analytic variety. More precisely, let $X(H)$ be the group of unramified character of $H(F)$ (a complex torus) and $ImX(H)$ the subgroup of unitary unramified characters. As a real group$ImX(H)$ is isomorphic to a product of $\mathbb{S}^1$. Hence, it has a natural structure of real analytic variety. Let $\{T_{ell}(H)\}$ be the set of orbits under this action. Each of these is the quotient of $ImX(M)$ by a finite subgroup and so is a real analytic variety.

We are now in position to define the spectral anlog of the space of (regular) conjugacy classes. Let

$$\mathcal{X}(H) = \left\{(M,\sigma); M \subset H \text{ Levi subgroup, } \sigma \in T_{ell}(M)\right\}/H(F) - conj$$

We may, and will, identify $\mathcal{X}(H)$ with a set of virtual tempered representations of $H(F)$ by

$$[M,\sigma] \in \mathcal{X}(H) \mapsto \pi = i_P^H(\sigma) \in R_{temp}(H)$$

for any choice of $P = MU_P$. We need also to define a measure on $\mathcal{X}(H)$. Fix a set $\mathcal{M}$ of representatives of the $H(F)$-conjugacy classes of Levi subgroup of H. Then we have

$$\mathcal{X}(H) = \bigsqcup_{M\in\mathcal{M}} T_{ell}(M)/W(M)$$

where $W(M)$ is the Weyl group of M. Each $T_{ell}(M)$ is a real analytic variety and so $\mathcal{X}(H)$ is naturally equipped with a topology. We put on $\mathcal{X}(H)$ the measure given by

$$\int_{\mathcal{X}(H)} f(\pi)d\pi = \sum_{M\in\mathcal{M}} \sum_{\mathcal{O}\in\{T_{ell}(M)\}} \int_{\mathcal{O}} f(i_M^H(\sigma))d\sigma$$

where the last measure $d\sigma$ is some $ImX(M)$-invariant measure on $\mathcal{O}$. We may now state the spectral expansion of Arthur's local trace formula

Theorem 3 *Let $\varphi, \varphi' \in C_c^\infty(H(F))$ and assume that φ is strongly cuspidal. Then, we have*

$$\lim_{N\to\infty} J_N(\varphi,\varphi') = \int_{\mathcal{X}(H)} (-1)^{a_{M(\pi)}} \Phi_{M(\pi)}(\pi,\varphi)\Phi_H(\pi^\vee,\varphi')d\pi$$

where $M(\pi)$ is any Levi from which π comes from and $\pi^\vee$ is the contragredient representation of π.

<u>**Remark**</u> : The distribution $\Phi_{M(\pi)}(\pi,\varphi)$ a priori depends on the choice of K as well as the choice of the date $(M(\pi),\sigma)$ from which π comes from. Actually, since φ is strongly cuspidal, it is possible to show that it doesn't depend on those. In particular, the RHS of the theorem makes sense.

Again, we will rewrite the result slightly differently. Set $\widehat{\theta}_\varphi(\pi) = (-1)^{a_{M(\pi)}}\Phi_{M(\pi)}(\pi,\varphi)$. We may now rewrite the spectral expansion as

$$(Spec) \quad \lim_{N\to\infty} J_N(\varphi,\varphi') = \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)\Phi_H(\pi^\vee,\varphi')d\pi$$

1.4.3 The final identity

Equaling the two expansions (Geom) and (Spec) of the limit, we are left with the identity

$$(Arthur) \quad \int_{H_{reg}(F)/conj} D^H(x)\theta_\varphi(x)\Phi_H(x,\varphi')dx = \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)\Phi_H(\pi^\vee,\varphi')d\pi$$

for all $\varphi, \varphi' \in C_c^\infty(H(F))$ with φ strongly cuspidal. By definition of the Harish-Chandra character, we have

$$\Phi_H(\pi^\vee,\varphi') = \int_{H(F)} \varphi'(h)\theta_{\pi^\vee}(h)dh$$

Plugging this in the spectral side of (Arthur), and switching the two integrals (the convergence is absolute), we obtain that the spectral side is equal to

$$\int_{H(F)} \varphi'(h) \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)\theta_{\pi^\vee}(h)d\pi dh$$

On the other hand, by the Weyl integration formula the geometric side is equal to

$$\int_{H(F)} \varphi'(h)\theta_\varphi(h)dh$$

Now, since the equality between the two ought to be true for all $\varphi' \in C_c^\infty(H(F))$, we obtain a sligthly different version of Arthur's local trace formula that will be the one we will use

$$(Arthur') \quad \theta_\varphi(h) = \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)\theta_{\pi^\vee}(h)d\pi$$

for all $h \in H_{reg}(F)$ and for all $\varphi \in C_c^\infty(H(F))$ that is strongly cuspidal.

Remark : It may be shown, and this will be use, that in (Arthur') the integrant is compactly supported (i.e. supported on a finite number of components. This due to the fact that for K' a compact-open subgroup of $H(F)$ there are only finitely many connected components of $\mathcal{X}(H)$ that contain representations having nonzero fixed vector by K'.

1.4.4 A first consequence

We will deduce here a first consequence of Arthur's local trace formula that will be of some use later. Let $\varphi \in C_c^\infty(H(F))$ be strongly cuspidal. Recall that the characters θ_π admit local expansions of the form

$$\theta_\pi(xe^X) = \sum_{\mathcal{O}\in Nil(\mathfrak{h}_x)} c_{\pi,\mathcal{O}}(x)\hat{j}(\mathcal{O}, X)$$

$x \in H_{ss}(F)$ and $\omega = \omega_\pi$ some open neighborhood of 0 in $\mathfrak{g}_x(F)$. It is easy to see that the neighborhood ω_π can be uniformly chosen for π in some compact subset of $\mathcal{X}(H)$. Moreover the coefficients $c_{\pi,\mathcal{O}}(x)$, seen as functions of π, are locally constant on $\mathcal{X}(H)$. Consequently, using (Arthur') and the remark at the end of the last paragraph, we deduce that θ_φ also admits local expansions of the same form :

$$\theta_\varphi(xe^X) = \sum_{\mathcal{O}\in Nil(\mathfrak{h}_x)} c_{\varphi,\mathcal{O}}(x)\hat{j}(\mathcal{O}, X)$$

where the coefficients $c_{\varphi,\mathcal{O}}(x)$ are given by

$$c_{\varphi,\mathcal{O}}(x) = \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)c_{\pi^\vee,\mathcal{O}}(x)d\pi$$

We may now define a function

$$c_\varphi : H_{ss}(F) \to \mathbb{C}$$

by

$$c_\varphi(x) = \frac{1}{\mid Nil_{reg}(\mathfrak{h}_x) \mid} \sum_{\mathcal{O}\in Nil_{reg}(\mathfrak{h}_x)} c_{\varphi,\mathcal{O}}(x)$$

We have the following formula for c_φ :

$$(1) \quad c_\varphi(x) = \int_{\mathcal{X}(H)} \widehat{\theta}_\varphi(\pi)c_{\pi^\vee}(x)d\pi$$

Call an invariant function $\theta : H(F) \to \mathbb{C}$ a quasicharacter if it admits local expansions of the form above. We may now restate our discussion by

Proposition 1 *For all strongly cuspidal $\varphi \in C_c^\infty(H(F))$, the function θ_φ is a quasicharacter.*

1.5 A local trace formula for GGP

Let's now consider a GGP triple (unitary case) :

$$G = U(V) \times U(W) \hookleftarrow H = N \rtimes U(W)$$
$$\xi : N(F) \to \mathbb{C}^\times$$

As before, we want to study the expression

$$J_N(f) = \int_{H(F)\backslash G(F)} \kappa_N(g) \int_{H(F)} f(g^{-1}hg)\xi(h)dhdg$$

$f \in C_c^\infty(G(F))$. We would like to give two expansions of this expression as $N \to \infty$: one geometric, the other one spectral. For that, we need to assume that f is strongly cuspidal.

We will first state the result, then explain why the formula for the multiplicity follows from it. Then I will try to give an overview of how the trace formula is proved.

1.5.1 The result

Recall that we have defined a space $\mathcal{C}(V, W)$ of conjugacy classes in $H(F)$:

$$\mathcal{C}(V, W) = \bigsqcup_{W' \subset W \ \text{nd}/U(W)(F)} U(W')(F)_{ell}/conj$$

and a measure on it.

For $f \in C_c^\infty(H(F))$ strongly cuspidal, we defined a function θ_f which is a quasicharacter. From this quasicharacter we constructed a function

$$c_f : G_{ss}(F) \to \mathbb{C}$$

by

$$c_f(x) = \frac{1}{|Nil_{reg}(\mathfrak{g}_x)|} \sum_{\mathcal{O} \in Nil_{reg}(\mathfrak{g}_x)} c_{f,\mathcal{O}}(x)$$

Now, extend the multiplicity $\pi \in Temp(G) \mapsto m(\pi)$ by linearity to $R_{temp}(G)$. We may now state the result

Theorem 4 *Let $f \in C_c^\infty(G(F))$ be strongly cuspidal. We have the following two expansions of the limit :*

$$(Geom) \quad \lim_{N\to\infty} J_N(f) = \lim_{s\to 0^+} \int_{\mathcal{C}(V,W)} D^H(x) c_f(x) \Delta(x)^s dx$$

$$(Spec) \quad \lim_{N\to\infty} J_N(f) = \int_{\mathcal{X}(G)} \widehat{\theta}_f(\pi) m(\pi^\vee) d\pi$$

How to deduce from this the formula for the multiplicity ? What we want is to prove that $m(\pi) = m_{geom}(\pi)$ for all $\pi \in R_{temp}(G)$. Recall that we have a decomposition

$$R_{temp}(G) = R_{ell}(G) \oplus R_{ind}(G)$$

and that $T_{ell}(G)$ is a basis of $R_{ell}(G)$. So it suffices to prove the formula separately for $\pi \in R_{ind}(G)$ and for $\pi \in T_{ell}(G)$

Case $\pi \in R_{ind}(G)$: Here we use induction. So we assume

(Hyp) The formula for the multiplicity is true for all pair $W' \subset V'$ of hermitian spaces (with odd codimension) such that

$$(dim\ V', dim\ W') \prec_{lex} (dim\ V, dim\ W)$$

where $\prec_{lex}$ denotes the lexical order.

By linearity we may assume that $\pi = i_Q^G(\sigma)$ where $Q = LU$ is a proper parabolic subgroup of G and $\sigma \in Temp(L)$. Without loss of generality, we may also assume that Q is a maximal proper subgroup of G. Consequently, we have either $Q = Q_V \times U(W)$ or $Q = U(V) \times Q_W$ where Q_V (resp. Q_W) is a maximal proper parabolic subgroup of $U(V)$ (resp. $U(W)$). We will treat the first case, the second one being analog. Since Q_V is a proper maximal subgroup of $U(V)$ there exists an isotropic subspace $Z \subset V$ such that

$$Q_V = Stab_{U(V)}(Z)$$

Let V' be an (automatically non degenerate) complement of Z in $Z^\perp$. Then Q_V admits a Levi decomposition $Q_V = L_V U_V$ where

$$L_V = GL(Z) \times U(V')$$

According to this decomposition, σ decomposes as a tensor product

$$\sigma = (\sigma_{GL} \otimes \sigma_{V'}) \otimes \sigma_W$$

Up to conjugation, we may assume that $V' \subset W$ or $W \subset V'$. This allows us in both cases to define the multiplicity $m(\sigma')$ and the geometric multiplicity $m_{geom}(\sigma')$ where $\sigma' = \sigma_{V'} \otimes \sigma_W \in Temp(G')$ and $G' = U(V') \times U(W)$. The formula for π now follows from the induction hypothesis once we have the following proposition

Proposition 2 *We have the equalities*

(i) $m(\pi) = m(\sigma')$ *;*

(ii) $m_{geom}(\pi) = m_{geom}(\sigma')$

Proof : I will only explain how to prove (ii). The character of π is supported on $L(F)^G = \{g^{-1}lg; g \in G(F)\ l \in L(F)\}$ where $L = L_V \times U(W)$ is a Levi component of Q. Hence the integral defining $m_{geom}(\pi)$ is in this case supported on $L(F)^G/conj \cap \mathcal{C}(V,W)$. It is easy to see that this intersection is contained in $G'(F) = U(V')(F) \times U(W)(F)$ and is equal to $\mathcal{C}(V',W)$ (seen as a space of conjugacy classes in $G'(F)$). Moreover, there is a well known relation between the character of σ and the character of π. Using this relation it is not so hard to obtain an analogous relation between c_π and c_σ. Applied here, it gives

$$D^H(x)c_\pi(x) = D^{H'}(x)c_{\sigma'}(x)c_{\sigma^{GL}}(1)$$

for all $x \in \mathcal{C}(V',W)$. Recall the result of Rodier saying that $c_{\sigma^{GL}}(1)$ counts the number of Whittaker models for σ^{GL}. Since σ^{GL} is a tempered representation of a general linear group, this number is one. Hence, $D^H(x)c_\pi(x) = D^{H'}(x)c_{\sigma'}(x)$ for all $x \in \mathcal{C}(V',W)$. We have now reduce the integral defining $m_{geom}(\pi)$ to the integral defining $m_{geom}(\sigma')$. ■

Case $\pi \in T_{ell}(G)$: Recall that we deduced from Arthur's trace formula that

$$c_f(x) = \int_{\mathcal{X}(G)} \widehat{\theta}_f(\pi)c_{\pi^\vee}(x)d\pi$$

Plugging this expression of c_f into the geometric side of (GGP), we get

$$(Geom) = \int_{\mathcal{X}(G)} \widehat{\theta}_f(\pi) \lim_{s\to 0^+} \int_{\mathcal{C}(V,W)} D^H(x)c_{\pi^\vee}(x)\Delta(x)^s dx d\pi$$

(we pushed the outside integral as well as the limit inside but this is justified by absolute convergence of the double integral and dominant convergence respectively). We recognize the term under the interior integral above : it is $m_{geom}(\pi^\vee)$. Now the identity between the geometric and the spectral expansions of the theorem may be written

$$(1) \quad \int_{\mathcal{X}(G)} \widehat{\theta}_f(\pi)\left(m(\pi^\vee) - m_{geom}(\pi^\vee)\right) d\pi = 0$$

for all strongly cuspidal $f \in C_c^\infty(G(F))$. Now, we have a natural decomposition

$$\mathcal{X}(G) = T_{ell}(G) \sqcup \mathcal{X}_{ind}(G)$$

where $\mathcal{X}_{ind}(G) = \mathcal{X}(G) \cap R_{ind}(G)$. Making use of the previous case, the previous equality becomes

$$\sum_{\pi \in T_{ell}(G)} \widehat{\theta}_f(\pi)m(\pi^\vee) = 0$$

It suffices now to find, for a given $\pi_0 \in T_{ell}(G)$, a strongly cuspidal function $f \in C_c^\infty(G(F))$ such that

$$\widehat{\theta}_f(\pi) = trace\ \pi(f) = \begin{cases} 1 & \text{if } \pi = \pi_0 \\ 0 & \text{otherwise.} \end{cases}$$

for all $\pi \in T_{ell}(G)$.

Proposition 3 *Such a strongly cuspidal function exists.*

The proof uses a Paley-Wiener theorem and a more general version of the local trace formula of Arthur.

1.5.2 Proof of the geometric expansion

Before going into its proof, recall the geometric expansion of the limit :

$$\lim_{N\to\infty} J_N(f) = \lim_{s\to 0^+} \int_{\mathcal{C}(V,W)} D^H(x) c_f(x) \Delta(x)^s dx$$

Remark one main difference with the geometric side of Arthur's local trace formula : in Arthur's formula you only see contributions from regular points whereas here a lot more semisimple conjugacy classes are contributing. These contributions reflect the existence of singularities in $J_N(\varphi)$ as $N \to \infty$. Therefore we cannot follow directly Arthur's proof which is unable to tell us what happens near these singularities. Indeed, let's try to follow Arthur's proof in the codimension one case. The first step is to apply the Weyl integration formula for H. As a result, we obtain the following expression of $J_N(f)$:

$$J_N(f) = \int_{H_{reg}(F)/conj} D^H(x) \int_{G_x(F)\backslash G(F)} f(g^{-1}xg)\kappa_{N,x}(g) dg dx$$

where

$$\kappa_{N,x}(g) = \int_{H_x(F)\backslash G_x(F)} \kappa_N(tg) dt$$

then Arthur proceeds to show that the integrand above as a limit which is a weighted orbital integral. The convergence of this integrand is uniform on compact subset of $H_{reg}(F)/conj$ so that Arthur's need to have a control of what happens near singular points. The main ingredient at this point of Arthur's proof is to show that the factor $D^H(x)$ take care of the possible divergence near singular points. For Arthur's triple, we have $D^H = (D^G)^{1/2}$ which is exactly what we need in order to control divergence around singular points. In the GGP case however, the factor D^H, which is usually smaller than $(D^G)^{1/2}$, cannot take care of that divergence. To get around this, we shall instead follow the following strategy :

1. Use a descent method imitated from Harish-Chandra to localize the problem near a semisimple point $x \in G_{ss}(F)$. Once the problem has been localize, we are left with an expression of the same type but for a localize triple (G_x, H_x, ξ_x) which is a product of a GGP triple and an Arthur triple. If x is not central the GGP triple is smaller than the one we started from and so we may use induction. So only the case where s is central needs further attention.
2. Near a central point we may use the exponential map to reduce everything over the Lie algebra ;
3. Over the Lie algebra we may perform a Fourier transform. After this Fourier transform, the expression converges well better and we are able to prove that the expression has a limit and to compute this limit (following the methods of Arthur).

Let's do the first step. For $x \in G_{ss}(F)$, we call slice through x an open $G_x(F)$-invariant neighborhood $\Omega \subset G_x(F)$ of x satisfying the following condition :

The map

$$\Omega \times_{G_x(F)} G(F) \to G(F)$$
$$(y, g) \mapsto g^{-1}yg$$

is an open immersion (where $\Omega \times_{G_x(F)} G(F) = (\Omega \times G(F))/G_x(F)$ the action being $(y,g).g_x = (g_x^{-1}yg_x, g_x^{-1}g))$

In particular, we have

$$\forall g \in G(F), \quad g^{-1}\Omega g \cap \Omega \neq \emptyset \Rightarrow g \in G_x(F)$$

By choosing Ω sufficiently small, we may assume that

$$D^G(y) = D^G(x) D^{G_x}(y), \quad \forall y \in \Omega$$

(where as usual D^G and D^{G_x} denote the absolute value of the Weyl determinants). Let $\Omega^G = \{g^{-1}yg; g \in G(F) \quad y \in \Omega\}$. Then using Weyl integration formula, we easily get

$$\int_{\Omega^G} \varphi(g) dg = D^G(x) \int_{G_x(F)\backslash G(F)} \int_{\Omega} \varphi(g^{-1}yg) dy dg$$

for all $\varphi \in C_c^\infty(\Omega^G)$.

There is also a notion of slice for the (usually nonreductive) group $H(F)$: for $x \in H_{ss}(F)$ a slice through x is an $H_x(F)$-invariant open neighborhood $\Omega_H \subset H_x(F)$ which is invariant by translation by $R_u(H_x)(F) = N_x(F)$ and such that

$$\Omega_H \times_{H_x(F)} H(F) \to H(F)$$
$$(y, h) \mapsto h^{-1}yh$$

is an open immersion. If the slice Ω_H is chosen sufficiently small, the same integration formula than before is true (with H instead of G).

The subsets of the form Ω^G ($x \in G_{ss}(F)$ and Ω a slice through x) form a basis for a topology on $G(F)$ (the "invariant topology"). We define exactly the same way an "invariant topology" on $H(F)$. The following two easy facts are left as exercises :

(A) $H(F)^G = \{g^{-1}hg; h \in H(F)\ g \in G(F)\}$ is closed in $G(F)$ equipped with the invariant topology ;

(B) The invariant topology on $H(F)$ is the same than the one induced from the invariant topology on $G(F)$ through the inclusion $H(F) \subset G(F)$.

Now, by some partition of unity, we may assume that there exists $x \in G_{ss}(F)$ and a slice $\Omega \subset G_x(F)$ through x sufficiently small such that

$$Supp(f) \subset \Omega^G$$

<u>Case $x \notin H(F)^G$</u> : Then by property (A) above, we may choose Ω sufficiently small such that $\Omega^G \cap H(F) = \emptyset$. Then we easily check that $J_N(f) = 0$ for all N and $J_{geom}(f) = 0$.

<u>Case $x \in H(F)^G$</u> : Then we may as well assume that $x \in H_{ss}(F)$. By property (B) above, we may choose Ω sufficiently small such that $\Omega^G \cap H(F) \subset \Omega_H^H$ for some small slice Ω_H through x in $H(F)$. If Ω_H is also sufficiently small, we may apply the descent integration formula and get

$$J_N(f) = D^H(x) \int_{H_x(F)\backslash G(F)} \kappa_N(g) \int_{\Omega_H} f(g^{-1}yg)\xi(y)dydg$$

For $g \in G(F)$, define ${}^g f_x \in C_c^\infty(H_x(F))$ to be given by

$${}^g f_x(h) = \begin{cases} f(g^{-1}hg) & \text{if } h \in \Omega_H \\ 0 & \text{otherwise.} \end{cases}$$

Then we may rewrite the previous expression of $J_N(f)$ as

$$J_N(f) = D^H(x) \int_{G_x(F)\backslash G(F)} J_{N,g}^{G_x}({}^g f_x)dg$$

where $J_{N,g}^{G_x}$ is the distribution on $G_x(F)$ defined by

$$J_{N,g}^{G_x}(\varphi) = \int_{H_x(F)\backslash G_x(F)} \kappa_N(g_x g) \int_{H_x(F)} \varphi(g_x^{-1}h_x g_x)\xi_x(h_x)dh_x dg_x$$

where $\xi_x = \xi_{|H_x(F)}$. Remark that this distribution is an analog for the triple (G_x, H_x, ξ_x) of the distribution J_N for the triple (G, H, ξ). Now I claim that the triple (G_x, H_x, ξ_x) is the product of an Arthur triple with a GGP triple. Indeed, let $W_x = Ker(x-1)_{|W}$, $V_x = Ker(x-1)_{|V}$ and $W^x = Im(x-1)$ then we have the decomposition

$$(G_x, H_x, \xi_x) = (U(V_x) \times U(W_x), N_x \rtimes U(W_x), \xi_x) \times (U(W^x) \times U(W^x), U(W^x), 1)$$

The first triple above is a GGP triple whereas the second one is an Arthur triple. Moreover, if x is central the GGP triple is smaller than the one we started from. By some induction argument and the local trace formula of Arthur we are left to consider only the case where x is central (I am swapping under the carpet the problem that ${}^g f_x$ is not always strongly cuspidal but there is a way to go around it). We may as well assume that $x = 1$, then assuming that $\Omega \subset G(F)$ is sufficiently small, we may assume that $\Omega = e^\omega$ where $\omega \subset \mathfrak{g}(F)$ is a $G(F)$-invariant open neighborhood of 0 on which the exponential map is defined and realized a measure-preserving isomorphism with an open subset of $G(F)$. Using the exponential map, we may then reduce everything over the Lie algebra. We get

$$J_N(f) = \int_{H(F)\backslash G(F)} \kappa_N(g) \int_{\mathfrak{h}(F)} f_\omega(g^{-1}Xg)\xi(X)dXdg$$

where

$$f_\omega(X) = \begin{cases} f(e^X) & \text{if } X \in \omega \\ 0 & \text{otherwise.} \end{cases}$$

and $\xi(X) = \xi(e^X)$. We now perform, as we said, a Fourier transform on the Lie algebra. Let $< .,. >$: $\mathfrak{g}(F) \times \mathfrak{g}(F) \to \mathbb{C}^\times$ be a $G(F)$-invariant symmetric nondegenerate bicharacter. This allows us, as we already saw, to define a Fourier transform :

$$\widehat{\varphi}(X) = \int_{\mathfrak{g}(F)} \varphi(Y) < X, Y > dY, \quad \varphi \in C_c^\infty(\mathfrak{g}(F))$$

such that $\widehat{\widehat{\varphi}}(X) = \varphi(-X)$. Denote by $\mathfrak{h}(F)^\perp$ the orthogonal of $\mathfrak{h}(F)$ in $\mathfrak{g}(F)$ wrt $< .,. >$. Let $\Xi \in \mathfrak{g}(F)/\mathfrak{h}(F)^\perp$ be the unique coset such that $\xi(X) =< \Xi, X >$, for all $X \in \mathfrak{h}(F)$. Set

$$\Sigma = \Xi + \mathfrak{h}(F)^\perp$$

Then it is an exercise to show the following Fourier inversion formula

$$\int_{\mathfrak{h}(F)} \varphi(X)dX = \int_\Sigma \widehat{\varphi}(Y)dY, \quad \varphi \in C_c^\infty(\mathfrak{g}(F))$$

(once the measures on $\mathfrak{h}(F)$ and Σ are chosen compatibly). Since we also have $\widehat{({}^g f_\omega)} = {}^g(\widehat{f_\omega})$, we get

$$J_N(f) = \int_{H(F)\backslash G(F)} \kappa_N(g) \int_\Sigma \widehat{f_\omega}(g^{-1}Yg)dYdg$$

Now, I claim that the following fact is true : there exists a (non empty) principal open subset $\Sigma' = \{Y \in \Sigma; Q(Y) \neq 0\}$, where $Q \in F[\mathfrak{g}]^G$, such that

– $\Sigma' \subset \mathfrak{g}_{reg}(F)$;
– The action by conjugation of $H(F)$ on Σ' is free ;
– Two elements of Σ' are $G(F)$-conjugate if and only if they are $H(F)$-conjugate.

From these properties, we get an injection $\Sigma'/H(F) \hookrightarrow \mathfrak{g}_{reg}(F)/conj$ and a measure $d_{\Sigma'/H}Y$ on $\Sigma'/H(F)$ such that

$$\int_{\Sigma} \varphi(Y)dy = \int_{\Sigma'/H(F)} \int_{H(F)} \varphi(h^{-1}Yh)dhd_{\Sigma'/H}Y$$

the last property we need is
– The injection $\Sigma'/H(F) \hookrightarrow \mathfrak{g}_{reg}(F)/conj$ is an open immersion sending the measure $d_{\Sigma'/H}Y$ to the measure $D^G(Y)^{1/2}d_{conj}Y$

So we may think of $\Sigma'/H(F)$ as an open subset of $\mathfrak{g}_{reg}(F)/conj$. Using these properties, we can transform the original expression $J_N(f)$ to

$$J_N(f) = \int_{\Sigma'/H(F)} D^G(Y)^{1/2} \int_{G_Y(F)\backslash G(F)} \widehat{f}_\omega(g^{-1}Yg)\kappa_{N,Y}(g)dgd_{conj}Y$$

where $\kappa_{N,Y}(g) = \int_{G_Y(F)} \kappa_N(tg)dt$. Remark that

$$f_\omega \text{ strongly cuspidal} \Rightarrow \widehat{f_\omega} \text{ strongly cuspidal.}$$

Now, we are in position to use the same method than Arthur, because the factor $D^G(Y)^{1/2}$ will take care of the divergence near singular point, and prove that

$$\begin{aligned}\lim_{N\to\infty} J_N(f) &= \int_{\Sigma'/H(F)} D^G(Y)^{-1/2} \int_{G_Y(F)\backslash G(F)} \widehat{f}_\omega(g^{-1}Yg)v_{M(Y)}(g)dgd_{conj}Y \\ &= \int_{\Sigma'/H(F)} D^G(Y)^{-1/2}\theta_{\widehat{f_\omega}}(Y)d_{conj}Y\end{aligned}$$

This proves that $J_N(f)$ has a limit and compute this limit. However we didn't get exactly the formula that we wanted. The very last step is to perform some inverse Fourier transform to get a formula involving f, and not $\widehat{f}$.

1.5.3 The Spectral Expansion

Now that we saw roughly how goes the proof of the geometric expansion, I would like to discuss briefly the proof of the spectral expansion. To simplify, I will assume that $m = d-1$ so that there is no unipotent part N and the character ξ is trivial. As Arthur for his local trace formula, the basic first step is to express f spectrally by mean of the Harish-Chandra-Plancherel formula :

$$f(x) = \int_{\mathcal{X}_0(G)} trace\left(\pi(x^{-1})\pi(f)\right)\mu(\pi)d\pi$$

for all $x \in G(F)$. Here $\mathcal{X}_0(G)$ is a space of tempered representations of $G(F)$ whose definition is very analogous to the one of $\mathcal{X}(G)$:

$$\mathcal{X}_0(G) := \{(M,\sigma); M \subset G \text{ Levi subgroup}, \ \sigma \in \Pi_2(G)\}/G(F) - conj$$

The measure $d\pi$ on $\mathcal{X}(G)$ is defined very similarly to the one on $\mathcal{X}(G)$ and $\mu(\pi)$ is the Harish-Chandra μ-function. Plugging this spectral expression of f in the definition of $J_N(f)$, we get

$$J_N(f) = \int_{H(F)\backslash G(F)} \kappa_N(g) \int_{\mathcal{X}_0(G)} \mathcal{L}_\pi(\pi(g)\pi(f)\pi(g^{-1})\mu(\pi)d\pi dg$$

where $\mathcal{L}_\pi : End(\pi)^\infty := \{T \in End(\pi); T \text{ is biinvariant by an open subgroup } K \subset G(F)\} \to \mathbb{C}$ is defined by

$$\mathcal{L}_\pi(T) = \int_{H(F)} trace\,(\pi(h)T)\,dh$$

(the integral is absolutely convergent). Obviously $\mathcal{L}_\pi$ is $H(F)\times H(F)$-invariant. Since $End(\pi)^\infty \simeq \pi \otimes \pi^\vee$ as a $G(F) \times G(F)$-representation, we obviously have

$$\mathcal{L}_\pi \neq 0 \Rightarrow m(\pi) \neq 0$$

The spectral expansion then follows closely the one of Arthur apart from one main point : we have to show the converse of the previous implication, namely

$$m(\pi) \neq 0 \Rightarrow \mathcal{L}_\pi \neq 0$$

Let me sketch the proof of that fact when π is square-integrable. So, assume there exists a nonzero element

$$\ell_\pi \in Hom_H(\pi, 1)$$

For $T \in End(\pi)^\infty$, we have

$$\int_{G(F)} trace\left(\pi(g^{-1})T\right)\pi(g)edg = d(\pi)^{-1}Te$$

for all $e \in V_\pi$ and where $d(\pi)$ is the formal degree of π (the integral is absolutely convergent). Now for $e \in V_\pi$ and $T \in End(\pi)^\infty$ there are two ways to compute the integral

$$\int_{G(F)} trace\left(\pi(g^{-1})T\right)\ell_\pi(\pi(g)e)dg$$

which is again absolutely convergent. The first way is to push the integral over $G(F)$ inside ℓ_π. By the above formula, we get

$$\int_{G(F)} trace\left(\pi(g^{-1})T\right)\ell_\pi(\pi(g)e)dg = d(\pi)^{-1}\ell_\pi(Te)$$

On the other hand, we may also decompose the integral as follows

$$\int_{G(F)} trace\left(\pi(g^{-1})T\right)\ell_\pi(\pi(g)e)dg = \int_{H(F)\backslash G(F)}\int_{H(F)} trace\left(\pi(g^{-1}h)T\right)dh\ell_\pi(\pi(g)e)dg$$
$$= \int_{H(F)\backslash G(F)} \mathcal{L}_\pi(T\pi(g^{-1}))\ell_\pi(\pi(g)e)dg$$

Choosing T and e such that $\ell_\pi(Te) \neq 0$, we see by the above two expansions that there is a $g \in G(F)$ such that $\mathcal{L}_\pi(T\pi(g^{-1})) \neq 0$, hence $\mathcal{L}_\pi \neq 0$.

2 A formula for certain ϵ-factors

In this section, we will see that there is a formula for certain ϵ-factors of pairs that is very similar to the one for the multiplicity discussed in the previous section. Even the proof of these formulas are very close. The main point here is to consider a twisted situation where the ϵ-factors will show up naturally. In the first paragraph, I collect some basic facts and definitions about twisted group. In the second part, I describe the particular twisted situation we want to study and I state a local trace formula for it. The resemblance between this twisted trace formula and the one developed in the previous section should be emphasize. On the spectral side we now have some "twisted" multiplicities. Exactly as before, our trace formula will naturally imply a formula for this twisted multiplicity. Last but not least, in the third paragraph we show that these twisted multiplicities are, in our particular situation, related to ϵ-factors of pairs.

2.1 Twisted Groups

We now need to consider a slight generalization of the abstract setting of triples (G, H, ξ) : we shall allow G and H to be twisted groups. Twisted groups have been introduce by Labesse and are a convenient way to talk about pairs (G, θ) where G is a connected algebraic group (over F) and θ is an automorphism of G.

Definition 2 *(i) A twisted group (over F) is a pair $(G, \widetilde{G})$ where*

- *G is a connected algebraic group over F ;*
- *$\widetilde{G}$ is a G-bitorsor, i.e. it is an algebraic variety over F with two left and right commuting actions*

$$G \times \widetilde{G} \times G \to \widetilde{G}$$
$$(g, \tilde{\gamma}, g') \mapsto g\tilde{\gamma}g'$$

each of them making $\widetilde{G}$ into a principal homogeneous space under G.

Remark *: For $(G, \widetilde{G})$ a twisted group, we have a natural action of $\widetilde{G}$ on G, $\widetilde{\gamma} \mapsto ad_{\widetilde{\gamma}}$, defined by the relation*

$$ad_{\widetilde{\gamma}}(g)\widetilde{\gamma} = \widetilde{\gamma}g$$

for all $g \in G$. Moreover, we may pass from a pair (G, θ) as above to a twisted group by setting $\widetilde{G} = G\theta$. Then we have $ad_\theta = \theta$.

We will always assume that our twisted groups satisfy the condition

$$\widetilde{G}(F) \neq \emptyset$$

In what follow,let $(G, \widetilde{G})$ be a twisted group (usually we will abbreviate and just talk about $\widetilde{G}$ to be a twisted group).

(ii) We say that the twisted group $\widetilde{G}$ is reductive if the connected group G is.

(iii) A (smooth) representation of $\widetilde{G}(F)$, is a pair $(\pi, \widetilde{\pi})$ where
- *$\pi : G(F) \to GL(V_\pi)$ is a smooth representation of $G(F)$;*
- *$\widetilde{\pi} : \widetilde{G}(F) \to GL(V_\pi)$ is a map such that*

$$\widetilde{\pi}(g\widetilde{\gamma}g') = \pi(g)\widetilde{\pi}(\widetilde{\gamma})\pi(g')$$

for all $g, g' \in G(F)$, $\widetilde{\gamma} \in \widetilde{G}(F)$.

Remark *: If $\widetilde{G}$ comes from a pair (G, θ) then a representation π of $G(F)$ extends to a representation $\widetilde{\pi}$ of $\widetilde{G}(F)$ if and only if $\pi \simeq \pi \circ \theta$.*

(iv) The (smooth) contragredient of a representation $(\pi, \widetilde{\pi})$ of $\widetilde{G}(F)$ is the representation $(\pi^\vee, \widetilde{\pi}^\vee)$ where
- *$\pi^\vee$ is the smooth contragradient of π ;*
- *$\widetilde{\pi}^\vee : \widetilde{G}(F) \to GL(V_{\pi^\vee})$ is the map defined by*

$$< \widetilde{\pi}(\widetilde{\gamma})v, \widetilde{\pi}^\vee(\widetilde{\gamma})v^\vee > = < v, v^\vee >$$

for all $v \in V_\pi$, $v^\vee \in V_{\pi^\vee}$.

(v) We say that a representation $(\pi, \widetilde{\pi})$ of $\widetilde{G}(F)$ is irreducible if π is.

(vi) Two representations $(\pi, \widetilde{\pi})$ and $(\pi', \widetilde{\pi}')$ of $\widetilde{G}(F)$ are said to be equivalent if $\pi \simeq \pi'$.

Remark *: We always have $\widetilde{\pi} \simeq c\widetilde{\pi}$ for all $c \in \mathbb{C}^\times$. We will denote by $Irr(\widetilde{G})$ the set of equivalence classes of irreducible representations $\widetilde{\pi}$ of $\widetilde{G}(F)$.*

(vii) If $\widetilde{G}$ is a reductive twisted group and $(\pi, \widetilde{\pi})$ is a representation of $\widetilde{G}(F)$ then we say that this representation is tempered (resp. square integrable) if the representation π is tempered (resp. square integrable). This allows us to define the corresponding subsets $Temp(\widetilde{G})$ and $\Pi_2(\widetilde{G})$ of $Irr(\widetilde{G})$.

We will use a slight abuse of notation and talk about a representation $\widetilde{\pi}$ of a twisted group $\widetilde{G}(F)$ the other component π being implicit.

Let $\widetilde{G}$ be a reductive twisted group and $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}(F)$. Then we may define as in the nontwisted case a distribution $\theta_{\widetilde{\pi}}$ on $\widetilde{G}(F)$ by setting

$$\theta_{\widetilde{\pi}}(\widetilde{f}) = trace\ \widetilde{\pi}(\widetilde{f})$$

for $\widetilde{f} \in C_c^\infty(\widetilde{G}(F))$. This distribution is, as in the nontwisted case, represented by a locally integrable function :

$$\theta_{\widetilde{\pi}}(\widetilde{f}) = \int_{\widetilde{G}(F)} \widetilde{f}(\widetilde{\gamma})\theta_{\widetilde{\pi}}(\widetilde{\gamma})d\widetilde{\gamma}$$

The function $\theta_{\widetilde{\pi}}$ (the character of $\widetilde{\pi}$) satisfies properties similar to characters of connected groups. In particular it admits local expansions

$$\theta_{\widetilde{\pi}}(\widetilde{x}e^X) = \sum_{\mathcal{O}\in Nil(\mathfrak{g}_{\widetilde{x}})} c_{\widetilde{\pi},\mathcal{O}}(\widetilde{x})\hat{j}(\mathcal{O}, X)$$

for all $X \in \omega$ a neighborhood of 0 in $\mathfrak{g}_{\widetilde{x}}(F)$ and where $\widetilde{x} \in \widetilde{G}_{ss}(F)$ is semisimple and $G_{\widetilde{x}} = Z_G(\widetilde{x})$ is the centralizer of $\widetilde{x}$ in G. Thus we may define, exactly the same way than in the connected case, a function

$$c_{\widetilde{\pi}} : \widetilde{G}_{ss}(F) \to \mathbb{C}$$

by averaging the coefficients of the previous local expansion corresponding to regular nilpotent orbits in $\mathfrak{g}_{\widetilde{x}}(F)$.

Let's now consider the following situation. Assume we have a triple $(\widetilde{G}, \widetilde{H}, \widetilde{\xi})$ where

- $(G, \widetilde{G})$ and $(H, \widetilde{H})$ are twisted groups over F the first one being reductive ;
- There is an embedding of twisted groups

$$(H, \widetilde{H}) \hookrightarrow (G, \widetilde{G})$$

- $\widetilde{\xi} : \widetilde{H}(F) \to \mathbb{C}^\times$ is a character of $\widetilde{H}(F)$ (i.e. a 1-dimensional representation).

Let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}(F)$. What is the analog of the multiplicity $m(\pi)$ in this situation ? We don't want to consider the space of $\widetilde{H}(F)$-equivariant homomorphisms

$$Hom_{\widetilde{H}}(\widetilde{\pi}, \widetilde{\xi})$$

because its dimension vary widely in the equivalence class of $\widetilde{\pi}$ (in which we may multiply $\widetilde{\pi}$ by a nonzero complex scalar) : in the same equivalence class it can be at the same time positive dimensional and zero. We shall give another definition for a multiplicity in this twisted setting. Consider the space

$$Hom_H(\pi,\xi)$$

and assume that it is finite dimensional. For $\ell \in Hom_H(\pi,\xi)$ and $\widetilde{h} \in \widetilde{H}(F)$ the linear form $\widetilde{\xi}(\widetilde{h})\ell \circ \widetilde{\pi}(\widetilde{h})^{-1}$ again belongs to $Hom_H(\pi,\xi)$ and doesn't depend on the choice of $\widetilde{h}$. Denote this new linear form by $\widetilde{\ell}$. Then we may define a "generalized" multiplicity $\epsilon(\widetilde{\pi})$ in our context by setting

$$\epsilon(\widetilde{\pi}) := trace\left(\ell \in Hom_H(\pi,\xi) \mapsto \widetilde{\ell}\right)$$

2.2 A local twisted trace formula

We will now introduce a case of the previous abstract setting where it is possible to develop as in the previous section a trace formula.

Fix for all nonnegative integer n an E-vector space V_n of dimension n with a basis $v_1,\ldots,v_n$ of it. Let $(M_n,\widetilde{M_n})$ be the following twisted group :

$$M_n := R_{E/F}GL(V_n)$$

$$\widetilde{M_n} := Sesq^*(V_n) := \{\widetilde{\gamma} : V_n \times V_n \to E \text{ nondegenerate sesquilinear form}\}$$

The right and left actions of M_n on $\widetilde{M_n}$ are given by

$$g\widetilde{\gamma}g' = \widetilde{\gamma}(g^{-1}.,g'.)$$

for all $g,g' \in M_n(F)$, $\widetilde{\gamma} \in \widetilde{M_n}(F)$.

Remark : We have an isomorphism $M_n \simeq R_{E/F}GL_n$ and the twisted group $(M_n,\widetilde{M_n})$ comes from the pair (M_n,θ_n) where θ_n is the automorphism of M_n given by $theta_n(g) = {}^t\overline{g}^{-1}$ (where $c : g \mapsto \overline{g}$ is conjugation with respect to the nontrivial element in $Gal(E/F)$). We deduce that a representation π admits an extension to $\widetilde{M_n}(F)$ iff $\pi \simeq \pi^\vee \circ c$. We will call such a representation a conjugate-dual representation.

We also need to fix embeddings, for $n \leqslant k$,

$$M_n \hookrightarrow M_k$$
$$\widetilde{M_n} \hookrightarrow \widetilde{M_k}$$

compatible in an obvious sense (that is embeddings of twisted groups). Since we fixed basis, we have a decomposition $V_k = V_n \oplus < v_{n+1},\ldots,v_k >$. This already gives a natural embedding $M_n \hookrightarrow M_k$. Now the embedding $\widetilde{M_n} \hookrightarrow \widetilde{M_k}$ is defined as follows : it sends $\widetilde{\gamma} \in \widetilde{M_n}$ to the sesquilinear form $\widetilde{\gamma} + h_{n,k}$, where $h_{n,k}$ is the hermitian form on $< v_{n+1},\ldots,v_k >$ defined by

$$h_{n,k}(v_i,v_j) = \delta_{i,j}$$

for $n+1 \leqslant i,j \leqslant k$.

We are now going to introduce the triples we are interested in. Let $d > m$ be two nonnegative integers of distinct parities : $d \not\equiv m \mod 2$. There is a triple associated to such a pair of integers and this is of the following form

$$\widetilde{G} = \widetilde{M_d} \times \widetilde{M_m} \hookleftarrow \widetilde{H} = \widetilde{M_m}$$

$$\widetilde{\xi} : \widetilde{H}(F) \to \mathbb{C}^\times$$
$$\widetilde{g}_m n \in \widetilde{H}(F) = \widetilde{M_m}(F) N(F) \mapsto \xi(n)$$

where N is a certain unipotent subgroup of M_d which is invariant by conjugation by $\widetilde{M_m}$ and ξ is a certain character on $N(F)$ invariant by conjugation by $\widetilde{M_m}(F)$. The embedding of $\widetilde{H}$ into $\widetilde{G}$ is given by the natural inclusion on the first factor and the natural projection onto $\widetilde{M_m}$ on the second factor. The unipotent subgroup N and its character ξ may be defined as follows. Let P be a minimal parabolic subgroup of M_d stable under conjugation by $\widetilde{M_m}$. Let $N = Rad_u(P)$ be its unipotent radical. Now choose ξ to be any $\widetilde{M_m}(F)$-invariant character of $N(F)$ which is generic for this property (i.e. with minimal stabilizer). We may give a more concrete description of (N,ξ) as follows. Recall that we have a decomposition $V_d = V_m \oplus < v_{m+1}, \ldots, v_d >$ and that we fixed an hermitian form $h_{d,m}$ on $<< v_{m+1}, \ldots, v_d >$. Since this spoace is odd dimensional, we may find another basis of it $(z_i)_{i=0,\pm1,\ldots,\pm r}$, where $2r+1 = d-m$, such that

$$h_Z(z_i, z_j) \neq 0 \Rightarrow i = -j$$

Let $Z_+ =< z_r, \ldots, z_1 >$. Now we may choose P to be the parabolic subgroup of M_d stabilizer of the flag

$$\begin{aligned} &< z_r > \subset \ldots \subset < z_r, \ldots, z_1 >= Z_+ \subset Z_+ \oplus < z_0 > \oplus V_m \\ &\qquad \subset Z_+ \oplus < z_0 > \oplus V_m \oplus < z_{-1} > \subset \ldots \subset Z_+ \oplus < z_0 > \oplus V_m \oplus < z_{-1}, \ldots, z_{-r} >= V_d \end{aligned}$$

In a basis of V_d adapted to this decomposition, P is the group of upper triangular by blocks matrix of some form. As we said, we take $N = Rad_u(P)$. We may now take for ξ the character of $N(F)$ defined by

$$\xi(n) = \psi\left(\sum_{i=-r}^{r-1} h_Z(nz_i, z_{-i-1})\right)$$

for all $n \in N(F)$, where $\psi : E \to \mathbb{C}^\times$ is a nontrivial character <u>trivial over F</u>.

We now would like to study the generalized multiplicity $\epsilon(\widetilde{\pi})$, $\widetilde{\pi} \in Temp(\widetilde{G})$, associated to the triple $(\widetilde{G}, \widetilde{H}, \widetilde{\xi})$ we just defined. For that, we are naturally lead to consider, as in the nonconnected case, a sequence of distributions of the form

$$J_N(\widetilde{f}) = \int_{H(F)\backslash G(F)} \kappa_N(g) \int_{\widetilde{H}(F)} \widetilde{f}(g^{-1}\widetilde{h}g)\widetilde{\xi}(\widetilde{h})d\widetilde{h}dg$$

for $\widetilde{f} \in C_c^\infty(\widetilde{G}(F))$. We would like this expression to have a limit as $N \to \infty$ and find two ways of computing the limit. As before, this is possible only for some particular choices of $(\kappa_N)_{N\geqslant 1}$ (but the hypothesis are still very loose and I won't discuss them) and we have to assume that $\widetilde{f}$ is strongly cuspidal (there is a natural notion of strongly cuspidal for twisted groups). With these restrictions, we are able to prove the existence of a limit and with a proof very similar to the one we already discussed, we obtain :

Theorem 5 *Let $\widetilde{f} \in C_c^\infty(\widetilde{G}(F))$ be strongly cuspidal. Then $J_N(\widetilde{f})$ admits a limit as $N \to \infty$ and we have the following two expansions of the limit :*

$$(Geom) \quad \lim_{N\to\infty} J_N(\widetilde{f}) = \lim_{s\to 0^+} \int_{\widetilde{\mathcal{C}}(d,m)} c_{\widetilde{f}}(\widetilde{x})\Delta(\widetilde{x})^s d\widetilde{x}$$

$$(Spec) \quad \lim_{N\to\infty} J_N(\widetilde{f}) = \int_{\mathcal{X}^{\widetilde{G}}} \widehat{\theta}_{\widetilde{f}}(\widetilde{\pi})\epsilon(\widetilde{\pi}^\vee)d\widetilde{\pi}$$

where

- *$\widetilde{\mathcal{C}}(d,m) = \bigsqcup_{n\leqslant n} \widetilde{M}_{n,ss}(F)/conj$ is a space of conjugacy classes in $\widetilde{G}(F)$ equipped with a natural measure on it ;*
- *$\mathcal{X}^{\widetilde{G}}$ is a certain space of virtual tempered representations of $\widetilde{G}(F)$, this is the twisted analog of the space $\widetilde{\mathcal{X}}^G$. It also comes with a natural measure on it.*
- *The other terms $c_{\widetilde{f}}$, $\widehat{\theta}_{\widetilde{f}}$, Δ are twisted versions of the analogous terms defined in the first section.*

For $\widetilde{\pi} \in Temp(\widetilde{G})$, define

$$\epsilon_{geom}(\widetilde{\pi}) := \lim_{s\to 0^+} \int_{\widetilde{\mathcal{C}}(d,m)} c_{\widetilde{\pi}}(\widetilde{x})\Delta(\widetilde{x})^s d\widetilde{x}$$

Exactly as with the GGP trace formula, we deduce from the previous theorem the following formula for the generalized multiplicity $\epsilon(\widetilde{\pi})$:

Corollary 1 *For all $\widetilde{\pi} \in Temp(\widetilde{G})$ we have the equality*

$$\epsilon(\widetilde{\pi}) = \epsilon_{geom}(\widetilde{\pi})$$

2.3 Relation between $\epsilon(\widetilde{\pi})$ and local ϵ factors of pairs

What will be interesting for us is that there exists a relation between the generalized multiplicity $\epsilon(\widetilde{\pi})$ and certain ϵ factors of pairs. Let $\widetilde{\pi} \in Temp(\widetilde{G})$ and write

$$\pi = \pi_d \otimes \pi_m$$

where $\pi_d \in Temp(M_d)$ and $\pi_m \in Temp(M_m)$ i.e. π_d and π_m are tempered representations of respectively $GL_d(E)$ and $GL_m(E)$ (up to the choice of basis). Jacquet, Piatetskii-Shapiro and Shalika have defined an L-function and an ϵ factor of pair

$$L(s, \pi_d \times \pi_m)$$

$$\epsilon(s, \pi_d \times \pi_m, \psi)$$

where ψ is a non trivial additive character of E. These are meromorphic functions of s of the form $P(q^{-s})^{-1}$, P a polynomial with $P(0) = 1$, and cq^{-ns} respectively. We are going to connect the generalized multiplicity $\epsilon(\widetilde{\pi})$ with the central value of the epsilon factor $\epsilon(1/2, \pi_d \times \pi_m, \psi)$. Let's denote this central value by $\epsilon(\pi, \psi)$. Remark that we cannot have a direct relation between the two : $\epsilon(\widetilde{\pi})$ is proportional to the normalization of $\widetilde{\pi}$ whereas $\epsilon(\pi, \psi)$ only depends on π but have a dependence in ψ. We shall rather connect $\epsilon(\pi, \psi)$ with the quotient of two generalized multiplicities. Since we are going to introduce a second generalized multiplicity for the twisted group $\widetilde{G}$ we will denote the one relative to $(\widetilde{H}, \widetilde{\xi})$ by $\epsilon_{GGP}(\widetilde{\pi})$ (because it is related to GGP). The other generalized multiplicity we are going to introduce will be denoted $\epsilon_{Whitt}(\widetilde{\pi})$ (because it is related to Whittaker models).

For all $n \geqslant 0$, set $\widetilde{h}_n = h_{0,n}$. So, $\widetilde{h}_n$ is the hermitian form on V_n defined by

$$\widetilde{h}_n(v_i, v_j) = \delta_{i,j}$$

Define θ_n to be the automorphism $ad_{\widetilde{h}_n}$ of M_n. Identifying M_n with $R_{E/F}GL_n$ via the basis $v_1, \ldots, v_n$, we have $g\theta_n = \theta_n {}^t\overline{g}^{-1}$ for all $g \in M_n$, where $g \mapsto \overline{g}$ is the conjugation with respect to the unique nontrivial F-automorphism of E. Denote by U_n the standard maximal unipotent subgroup of M_n. Also, set

$$w_n = ...$$

It is easy to check that $w_n\theta_n$ fixes U_n. Let $\psi : E \to \mathbb{C}^\times$ be a nontrivial additive character that is trivial over F. We may define a generic character ψ_n of $U_n(F)$ by

$$\psi_n(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$$

for $u \in U_n(F)$, where $u_{i,j}$ denote the coefficients of the matrix of u in the basis $v_1, \ldots, v_n$. We check that $w_n\theta_n$ fixes the character ψ_n. Now set

$$\widetilde{U}_n = U_n w_n \theta_n$$

$$\widetilde{\psi}_n : \widetilde{U}_n(F) \to \mathbb{C}^\times$$
$$\widetilde{\psi}_n(uw_n\theta_n) = \psi_n(u)$$

So $\widetilde{U}_n$ is a twisted group (under U_n) and $\widetilde{\psi}_n$ is a character of $\widetilde{U}_n(F)$. Returning to the twisted group $\widetilde{G} = \widetilde{M}_d \times \widetilde{M}_m$, we may now define its twisted subgroup

$$\widetilde{U} = \widetilde{U}_d \times \widetilde{U}_m$$

and a twisted character on it

$$\widetilde{\psi} = \widetilde{\psi}_d \otimes \widetilde{\psi}_m$$

This allows us to define a new generalized multiplicity $\epsilon_{Whitt}(\widetilde{\pi})$, $\widetilde{\pi} \in Temp(\widetilde{G})$, with respect to the pair $(\widetilde{U}, \widetilde{\psi})$. We have now the following

Proposition 4 *For all $\widetilde{\pi} \in Temp(\widetilde{G})$, we have an equality*

$$\epsilon_{GGP}(\widetilde{\pi}) = \epsilon_{Whitt}(\widetilde{\pi})\epsilon(\pi, \psi)\omega_{\pi_m}(-1)$$

Remark : Although not apparent in the notation, the generalized multiplicity $\epsilon_{Whitt}(\widetilde{\pi})$ depends on the character ψ. This cancels on the RHS the dependence in ψ of $\epsilon(\pi, \psi)$.

Proof : This follows rather directly from the local functional equation of Jacquet, Piatetskii-Shapiro and Shalika. Let me explain how it works in the case $m = d - 1$. Decompose $\pi = \pi_d \otimes \pi_m$, $\pi_d \in Temp(M_d)$, $\pi_m \in Temp(M_m)$. Fix two nonzero Whittaker functionals

$$\lambda_{\pi_d} \in Hom_{U_d}(\pi_d, \psi_d)$$
$$\lambda_{\pi_m} \in Hom_{U_m}(\pi_m, \psi_m)$$

(we may find some because π_d and π_m are tempered). Let

$$e \in V_{\pi_d} \mapsto W_e^d \in C_c^\infty(U_d(F)\backslash M_d(F), \psi_d)$$
$$e' \in V_{\pi_m} \mapsto W_{e'}^m \in C_c^\infty(U_m(F)\backslash M_m(F), \psi_m)$$

be the corresponding Whittaker models. So $W_e(g) = \lambda_{\pi_d}(\pi_d(g)e)$ for $e \in V_{\pi_d}$ and $W_{e'}(h) = \lambda_{\pi_m}(\pi_m(h)e')$ for $e' \in V_{\pi_m}$.

Remark now that $Hom_U(\pi, \psi_d \otimes \psi_m)$ is one dimensional and has a basis given by $\lambda_d \otimes \lambda_m$. By definition of the generalized multiplicity, we have

$$\epsilon_{Whitt}(\widetilde{\pi})(\lambda_d \otimes \lambda_m) = (\lambda_d \otimes \lambda_m) \circ \widetilde{\pi}(w_d\theta_d, w_m\theta_m)$$

By homogeneity of the equality of the proposition, we may assume that $\widetilde{\pi}$ is normalized so that $\epsilon_{Whitt}(\widetilde{\pi}) = 1$. This means that $\widetilde{\pi} = \widetilde{\pi}_d \otimes \widetilde{\pi}_m$ where $\widetilde{\pi}_d$ and $\widetilde{\pi}_m$ are the unique extensions of π_d and π_m to $\widetilde{M}_d(F)$ and $\widetilde{M}_m(F)$ respectively such that

$$\lambda_{\pi_d} \circ \widetilde{\pi}_d(w_d\theta_d) = \lambda_{\pi_d}$$
$$\lambda_{\pi_m} \circ \widetilde{\pi}_m(w_m\theta_m) = \lambda_{\pi_m}$$

For $s \in \mathbb{C}$, $e \in V_{\pi_d}$ and $e' \in V_{\pi_m}$, let

$$\mathcal{L}(W^d_{e'}, W^m_e, s) = \int_{U_m(F)\backslash M_m(F)} W^m_{e'}(a_m h) W^d_e(h) |det\ h|^{s-1/2} dh$$

where a_m is a diagonal matrix with diagonal coefficients that are alternating signs. Jacquet, Piatetskii-Shapiro and Shalika have proved the following properties :

(1) this integral is absolutely convergent for $Re(s) > 0$ and can be extended as a meromorphic function of s on the whole complex plane ;
(2) we can choose e and e' such that $\mathcal{L}(W^m_{e'}, W^d_e, 1/2) \neq 0$;
(3) setting $\check{W}_e(g) = W_e(w_d{}^t\overline{g}^{-1})$ and $\check{W}_{e'}(h) = W_{e'}(w_m{}^t\overline{h}^{-1})$, we have the functional equation

$$\mathcal{L}(\check{W}_{e'}, \check{W}_e, 1-s)/L(1-s, \pi_d \times \pi_m) = \omega_{\pi_m}(-1)\epsilon(s, \pi_d \times \pi_m, \psi)\mathcal{L}(W_{e'}, W_e, s)/L(s, \pi_d \times \pi_m)$$

for all $e \in V_{\pi_d}$, $e' \in V_{\pi_m}$ and $s \in \mathbb{C}$ (I simplified the functional equation using the fact that π_d and π_m are conjugate-dual).

Now observe that the linear form

$$e \otimes e' \in V_\pi = V_{\pi_d} \otimes V_{\pi_m} \mapsto \ell(e \otimes e') = \mathcal{L}(W^d_e, W^m_{e'}, 1/2)$$

is $M_m(F) = H(F)$-invariant i.e. it belongs to $Hom_H(\pi, 1)$. By property (2) above this linear form is nonzero. On the other hand, by a result of Aizenbud, Gourevitch, Rallis and Schiffmann the space $Hom_H(\pi, 1)$ is at most one dimensional. Consequently, ℓ is a basis of $Hom_H(\pi, 1)$ and we have, by definition of the generalized multiplicity,

$$\ell \circ \widetilde{\pi}(\widetilde{h}) = \epsilon_{GGP}(\widetilde{\pi})\ell$$

for all $\widetilde{h} \in \widetilde{H}(F)$. We may choose $\widetilde{h}$ to be $\theta_m \in \widetilde{M}_m(F) = \widetilde{H}(F)$. The previous relation is then equivalent to

$$(4) \quad \mathcal{L}(W^m_{e'_1}, W^d_{e_1}, 1/2) = \epsilon_{GGP}(\widetilde{\pi})\mathcal{L}(W^m_{e'}, W^d_e, 1/2)$$

for all $e' \in V_{\pi_m}$, $e \in V_{\pi_d}$ and where $e'_1 = \widetilde{\pi}_m(\theta_m)e'$ and $e_1 = \widetilde{\pi}_d(\theta_d)e$ (the image of θ_m in $\widetilde{M_d}$ is θ_d). Because λ_{π_d} is invariant by $\widetilde{\pi}(w_d\theta_d)$, we have

$$\begin{aligned} W^d_{e_1}(g) &= \lambda_{\pi_d}\left(\pi(g)\widetilde{\pi}(\theta_d)e\right) = \lambda_{\pi_d}\left(\widetilde{\pi}(\theta_d)\pi({}^t\overline{g}^{-1})e\right) \\ &= \lambda_{\pi_d}\left(\pi(w_d^2)\widetilde{\pi}(\theta_d)\pi({}^t\overline{g}^{-1})e\right) = \lambda_{\pi_d}\left(\widetilde{\pi}(w_d\theta_d)\pi(w_d{}^t\overline{g}^{-1})e\right) \\ &= \lambda_{\pi_d}\left(\pi(w_d{}^t\overline{g}^{-1})e\right) = \check{W}^d_e(g) \end{aligned}$$

Similarly, we have $W^m_{e'_1} = \check{W}^m_{e'}$. Now comparing (4) with the functional equation (3) evaluated at $s = 1/2$ (the L-function has no pole nor zero at $s = 1/2$) immediately lead to the relation of the proposition. ■