The local Gan Gross Prasad conjecture lecture 4

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3 Endoscopy and the more refined GGP conjecture

In this last lecture, I will state the more precise version of GGP and sketch a proof of it. The crucial ingredients are of course the two integral formulas we have been discussing (for the multiplicity and the ϵ-factors). These two will be connected using the theory of endoscopy. Actually, we will also need endoscopy to state a precise version of the local Langlands correspondence (for unitary groups). So, we will first discuss the more precise version of LLC as well as its endoscopic characterization. After that, we will be able to state the more refined version of GGP and sketch a proof of it.

We fix once and for all a nontrivial additive character $\psi : E \to \mathbb{C}^\times$ which is trivial over F. This character will mainly serve to normalize the local Langlands correspondence.

3.1 Local Langlands correspondence for unitary groups

3.1.1 Langlands parameters

Before going into the statement of the conjecture, I would like to discuss the relevant Langlands parameters. Recall that for G a reductive connected group over F, a Langlands parameter is an homomorphism

$$\varphi : WD_F \to {}^L G$$

where $WD_F = W_F \times SU(2,\mathbb{R})$ is the Weil-Deligne group of F and ${}^L G = \widehat{G} \rtimes W_F$ is the Langlands dual group of G. This parameter has to satisfy some properties : it has to be continuous, semisimple and make the following diagram commute

$$\begin{tikzcd} WD_F \arrow[r] \arrow[d] & {}^L G \arrow[d] \\ W_F \arrow[r, equal] & W_F \end{tikzcd}$$

Moreover, two Langlands parameters are equivalent if they are conjugated under $\widehat{G}$.

Let $(G, \widetilde{G})$ be a twisted group over F. We also have a notion of L-group and Langlands parameter in this twisted situation : the L-group is a twisted group ${}^L\widetilde{G}$ under ${}^LG$ and a Langlands parameter for $\widetilde{G}$ is a pair $(\varphi, \widetilde{\varphi})$ of maps

$$\varphi : WD_F \to {}^LG$$
$$\widetilde{\varphi} : WD_F \to {}^L\widetilde{G}$$

such that φ is a Langlands parameter for G and $\widetilde{\varphi}$ satisfy

$$\widetilde{\varphi}(ww') = \varphi(w)\widetilde{\varphi}(w') = \widetilde{\varphi}(w)\varphi(w')$$

for all $w, w' \in WD_F$. There is also a natural notion of equivalence for these more general Langlands parameters.

A Langlands parameter for a group G or a twisted group $\widetilde{G}$ is expected to parametrize a packet of tempered representations if this parameter is tempered meaning that its image in $\widehat{G}$ is bounded. We will be uniquely interested in tempered Langlands parameters for unitary groups $U(V)$ and the twisted groups $\widetilde{M}_n$. In both these cases, the (tempered) Langlands parameters may be described more concretely in terms of some Weil-Deligne representations. More precisely, Langlands parameters in these contexts are given by certain complex representations of WD_E (the Weil-Deligne group of E) satisfying some duality properties (rather conjugate-duality properties). In order to describe those, it is convenient to fix an element $\sigma \in W_F \backslash W_E$ (so that σ restricts to the nontrivial F-automorphism on E). Then σ acts by conjugation on WD_E and we will denote by $w \mapsto \sigma w \sigma^{-1} = w^\sigma$ the corresponding automorphism.

Definition 1 *Let $\varphi : WD_E \to GL(M)$ be a complex finite-dimensional representation of WD_E that is continuous and semisimple. Then :*

(i) We say that φ is conjugate-dual if there exists a nondegenerate bilinear form $B : M \times M \to \mathbb{C}$ such that

$$B(\varphi(w)e, \varphi(w^\sigma)e') = B(e, e')$$

for all $e, e' \in M$, $w, w' \in WD_E$. Such a form B is called a conjugate-dual form on φ.

(ii) Let $\epsilon \in \{\pm\}$ be a sign. We say that φ is ϵ-conjugate-dual if there exists a nondegenerate bilinear form $B : M \times M \to \mathbb{C}$ satisfying the previous condition as well as

$$B(e, \varphi(\sigma^2)e') = \epsilon B(e, e')$$

for all $e, e' \in M$. Such a form B is called a ϵ-conjugate-dual form on φ.

Remark : Let $\mu : WD_E \to \mathbb{C}^\times$ be a character of the Weil-Deligne group of E. By local class field theory, μ is identified with a character of $E^\times$. Using this identification we have the following : μ is conjugate dual iff μ is trivial on $N(E^\times)$ and μ is conjugate-dual of sign $+$ (resp. of sign $-$) iff $\mu_{|F^\times} = 1$ (resp. iff $\mu_{|F^\times} = \eta_{E/F}$). Here $\eta_{E/F} : F^\times \to F^\times/N(E^\times) \simeq \{\pm 1\}$ is the quadratic character associated to the quadratic extension E/F.

Let us first introduce the tempered Langlands parameters for $\widetilde{M}_n$, $n \geqslant 0$. Such a Langlands parameter is just a conjugate-dual representation $\varphi : WD_E \to GL(M)$ of dimension n (continuous and semisimple) with bounded image. We will denote by $\widetilde{\Phi}_{temp}(n)$ the set of such representations taken upon isomorphism. This is the set of tempered Langlands parameters for $\widetilde{M}_n$. Remark that the local Langlands correspondence is known in this case and is obtained as follows. Start with a Langlands parameter $\varphi \in \widetilde{\Phi}_{temp}(n)$. In particular, φ is an n-dimensional representation of WD_E. By the local Langlands correspondence for GL_n, established by Harris-Taylor, Henniart and Scholze, this corresponds a tempered representation $\pi(\varphi)$ of $GL_n(E) \simeq M_n(F) : \pi(\varphi) \in Temp(M_n)$. Since the parameter φ is conjugate-dual, so is the representation $\pi(\varphi)$ i.e. $\pi(\varphi)$ admits an extension $\widetilde{\pi}(\varphi)$ to $\widetilde{M}_n(F)$. The local Langlands correspondence for $\widetilde{M}_n$ is now the map that associates to $\varphi \in \widetilde{\Phi}_{temp}$ the equivalence class of $\widetilde{\pi}(\varphi)$ (Note that $\widetilde{\pi}(\varphi)$ is only well defined up to a constant).

Let us now deal with unitary groups. So let V be an hermitian space of dimension n and $U(V)$ its unitary group. A tempered Langlands parameter for $U(V)$ is a $(-1)^{n+1}$-conjugate-dual representation $\varphi : WD_E \to GL(M)$ of dimension n (continuous and semisimple) with bounded image. We will denote by $\Phi_{temp}(n)$ the set of isomorphism classes of such representations. So $\Phi_{temp}(n)$ is our set of tempered Langlands parameters for $U(V)$. Although not necessarily unique, it is sometimes convenient to assume that a parameter $\varphi \in \Phi_{temp}(n)$ comes equipped with a particular $(-1)^{n+1}$-conjugate-dual form B on it. Then, we will set

$$A_\varphi = \pi_0(Aut(\varphi, B))$$

It is an elementary 2-abelian finite group. Let $z_\varphi \in A_\varphi$ be the image of $-I$. The pair (A_φ, z_φ) actually doesn't depend on B (up to an unique isomorphism), it is why we don't need to carry B in the notation.

3.1.2 The Local Langlands Correspondence for unitary groups

The local Langlands correspondence is better expressed when considering two unitary groups at the same time. Let V_i and V_a be two representatives of the isomorphism classes of hermitian spaces of dimension n with $U(V_i)$ quasisplit. Then the LLC states two things. First there should exist a decomposition

$$Temp(U(V_i)) \sqcup Temp(U(V_a)) = \bigsqcup_{\varphi \in \Phi_{temp}(n)} \Pi(\varphi)$$

into finite sets called L-packets. So in particular an L-packet $\Pi(\varphi)$ admits a decomposition $\Pi(\varphi) = \Pi_i(\varphi) \sqcup \Pi_a(\varphi)$ where $\Pi_\natural(\varphi) = \Pi(\varphi) \cap Temp(U(V_\natural))$ ($\natural = i$ or a). Secondly, there should exist bijections

$$\widehat{A}_\varphi \simeq \Pi(\varphi)$$
$$\chi \mapsto \pi(\varphi, \chi)$$

where $\widehat{A}_\varphi := \{\chi : A_\varphi \to \{\pm 1\}\}$ is the set of characters of A_φ. The bijection should be such that $\pi(\varphi, \chi) \in \Pi_i(\varphi)$ (resp. $\pi(\varphi, \chi) \in \Pi_a(\varphi)$) iff $\chi(z_\varphi) = 1$ (resp. $\chi(z_\varphi) = -1$). Of course, we ask this parametrization to satisfy some properties. The properties we will impose are of endoscopic nature. Actually, we will see that our assumptions determine the parametrization uniquely (if it exists). A representation is uniquely determined by its character θ_π and these characters are linearly independent. So, by basic Fourier inversion, it is sufficient to understand the functions

$$\theta_{\varphi,s} = \sum_{\chi \in \widehat{\mathcal{S}}_\varphi} \chi(s) \theta_{\pi(\varphi,\chi)}$$

seen as a function on $U(V_i)_{reg}(F) \sqcup U(V_a)_{reg}(F)$, for all $\varphi \in \Phi_{temp}(n)$ and $s \in \mathcal{S}_\varphi$. We will denote by $\theta^\natural_{\varphi,s}$ the restriction of $\theta_{\varphi,s}$ to $U(V_\natural)_{reg}(F)$ ($\natural = i$ or a). The first condition we impose is the following

(STAB) For all $\varphi \in \Phi_{temp}(n)$, the character $\theta^i_\varphi = \theta^i_{\varphi,0}$ is stable (i.e. is constant on stable conjugacy classes).

The heart of the endoscopic characterization of LLC is now to come. Using classical endoscopy first, we will express the character $\theta_{\varphi,s}$ in terms of stable characters $\theta^i_{\varphi'}$ leaving on smaller quasisplit unitary groups. Using then a twisted endoscopy, we will express the stable characters θ^i_φ in terms of twisted characters over the twisted group $\widetilde{M}_n(F)$. Taken together, we obtain an expression of $\theta_{\varphi,s}$ in terms of twisted characters. This will specify uniquely, if it exists, the local Langlands correspondence.

3.1.3 Classical Endoscopy

The elliptic endoscopic groups of $U(V)$ ($V = V_i$ or V_a) are of the form $U(V_{i,+}) \times U(V_{i,-})$ where $V_{i,\pm}$ are quasisplit hermitian spaces of dimensions $n_\pm$ such that $n = n_+ + n_-$. There is an underlying endoscopic datum which among other things provides us with an embedding of L-groups

$${}^L\left(U(V_{i,+}) \times U(V_{i,-})\right) \hookrightarrow {}^L U(V)$$

which depends on the choice of two conjugate-dual characters $\mu_+, \mu_- : WD_E \to \mathbb{C}^\times$ of sign $(-1)^{n_-}$ and $(-1)^{n_+}$ respectively. At the level of Langlands parameters, the previous homomorphism of L-groups induces a map (by push-forward)

$$\Phi_{temp}(n_+) \times \Phi_{temp}(n_-) \to \Phi_{temp}(n)$$

which is given by

$$(\varphi_+, \varphi_-) \mapsto \mu_+\varphi_+ \oplus \mu_-\varphi_-$$

Remark that without the twists by μ_+ and μ_- we wouldn't get a map into $\Phi_{temp}(n)$ because the sign of the conjugate-dual representation $\varphi_+ \oplus \varphi_-$ wouldn't match. On the other hand, we also get from the endoscopic datum a correspondence

$$U(V_{i,+})_{reg}(F)/stab \times U(V_{i,-})_{reg}(F)/stab \longleftrightarrow U(V)_{reg}(F)/stab$$

which may be explicitely described as follows : $(y_+, y_-) \in U(V_{i,+})_{reg}(F) \times U(V_{i,-})_{reg}(F)$ and $x \in U(V)_{reg}(F)$ correspond to each other iff $y = y_+ + y_- \in U(V')(F)$ is regular ($V' = V_{i,+} \oplus V_{i,-}$) and is stably conjugate to x. Recall that since V' and V have same dimension, there is an isomorphism $U(V)(\overline{F}) \simeq U(V')(\overline{F})$ well defined up to conjugation and which allow us to make sense of the statement : y and x are stably conjugate.

From the endoscopic datum we may deduce transfer factors. These is a function

$$\Delta : (U(V_{i,+})_{reg}(F)/stab \times U(V_{i,-})_{reg}(F)/stab) \times U(V)_{reg}(F)/conj \to \mathbb{C}$$

$$((y_+, y_-), x) \mapsto \Delta(y_+, y_-, x)$$

such that $\Delta(y_+, y_-, x) \neq 0$ iff (y_+, y_-) and x correspond to each other. However, these transfer factors are only well defined once you make an additional choice : that of a Whittaker datum for $U(V_i)$. If n is odd, this choice is harmless because there is only one conjugacy class of Whittaker datum. However, if n is even there are two conjugacy classes of such Whittaker datum and a choice has to be made. I claim that the additive character ψ we made allows us to pick a particular Whittaker datum. This is the one we choose. Because of this dependence in ψ, we will denote our transfer factors by Δ_ψ (although there is no dependence if n is odd).

Assume given two functions

$$\theta : U(V)_{reg}(F)/conj \to \mathbb{C}$$

and

$$\theta' : U(V_{i,+})_{reg}(F)/stab \times U(V_{i,-})_{reg}(F)/stab \to \mathbb{C}$$

We will say that θ is the transfer of θ' if

$$\theta(x) = \sum_y \Delta_\psi(y, x)\theta'(y)$$

for all $x \in U(V)_{reg}(F)$, and where the sum is over all stable conjugacy classes in $U(V_{i,+})_{reg}(F) \times U(V_{i,-})_{reg}(F)$.

We may now state our second condition about LLC. Let $(\varphi_+, \varphi_-) \in \Phi_{temp}(n_+) \times \Phi_{temp}(n_-)$ and denote by $\varphi = \mu_+\varphi_+ \oplus \mu_-\varphi_- \in \Phi_{temp}(n)$ the corresponding Langlands parameter of

$U(V)$. There is also a element $s \in A_\varphi$ coming from this decomposition. Namely, choose a $(-1)^{n_++1}$-conjugate-dual form B_+ on φ_+ and a $(-1)^{n_-+1}$-conjugate-dual form B_- on φ_-. Then $B = B_+ \oplus B_-$ is a $(-1)^{n+1}$-conjugate-dual form on φ and we may pick the element s of $Aut(\varphi, B)$ that acts trivially on $\mu_+\varphi_+$ and by $-I$ on $\mu_-\varphi_-$. The image of s in A_φ doesn't depend on the choices of B_+, B_- and will also be denoted by s. The second condition is now the following

(CE) The characters $\theta^i_{\varphi,s}$ and $(-1)^{n+1}\theta^a_{\varphi,s}$ are transfers of $\theta^i_{\varphi_+} \times \theta^i_{\varphi_-}$.

Since all element $s \in \mathcal{S}_\varphi$ may be obtained in this manner, this reduces the determination of the characters $\theta_{\varphi,s}$ to the one of the form θ^i_φ (but for different Langlands parameters φ).

3.1.4 Twisted Endoscopy

There is also a twisted theory of endoscopy. The group $U(V_i)$ is an elliptic twisted endoscopic group of $\widetilde{M}_n$ ($dim(V_i) = n$). There is an underlying endoscopic datum which among other things provides us with an embedding of twisted L-groups

$$^LU(V_i) \hookrightarrow {}^L\widetilde{M}_n$$

where $^LU(V_i)$ is considered as a trivial twisted group under itself. This embedding depends on the choice of a conjugate-dual character $\mu : WD_E \to \mathbb{C}^\times$. At the level of Langlands parameters, the previous embedding gives rise to a map (by push-forwards)

$$\Phi_{temp}(n) \to \widetilde{\Phi}_{temp}(n)$$

which is just $\varphi \mapsto \mu\varphi$. We also get from the endoscopic datum a correspondence

$$U(V_i)_{reg}(F)/stab \longleftrightarrow \widetilde{M}_{n,reg}(F)/stab$$

which is only well-defined once we choose a particular element of $\widetilde{M}_n(F)$. We may choose the element $w_n\theta_n$ defined in the last lecture. We also get transfer factors

$$\Delta : U(V_i)_{reg}(F)/stab \times \widetilde{M}_{n,reg}(F)/conj \to \mathbb{C}$$

such that $\Delta(y, \widetilde{x}) \neq 0$ if and only if y and $\widetilde{x}$ correspond to each other. As before, there is an additional choice to be made : that of a Whittaker datum of M_n stable by $w_n\theta_n$. We will choose the Whittaker datum (U_n, ψ_n) defined last time. Again this choice depends on the additive character ψ we fixed.

Assume given two functions

$$\widetilde{\theta} : \widetilde{M}_{n,reg}(F)/conj \to \mathbb{C}$$

and

$$\theta : U(V_i)_{reg}(F)/stab \to \mathbb{C}$$

We will say that $\widetilde{\theta}$ is the transfer of θ if

$$\widetilde{\theta}(\widetilde{x}) = \sum_y \Delta_\psi(y, \widetilde{x})\theta(y)$$

for all $\widetilde{x} \in \widetilde{M}_{n,reg}(F)$, and where the sum is over all stable conjugacy classes in $U(V_i)_{reg}(F)$. Since in this case the correspondence between stable regular conjugacy classes is a bijection, you may also see the last formula as a way to express θ in terms of $\widetilde{\theta}$.

We are now in position to state the third and last property of the local Langlands correspondence we need. Let $\varphi \in \Phi_{temp}(n)$. This Langlands parameter maps to a Langlands parameter $\mu\varphi$ of $\widetilde{\Phi}_{temp}(n)$. By the local Langlands correspondence for $\widetilde{M}_n$, $\mu\varphi$ determines an equivalence class of tempered representations $\widetilde{\pi}(\mu\varphi)$ of $\widetilde{M}_n(F)$. We may pick one distinguished element $\widetilde{\pi}(\varphi)$ in this equivalence class by asking that $\epsilon_{Whitt}(\widetilde{\pi}(\mu\varphi)) = 1$ (this normalization also depends on the character ψ). The last condition we impose on the local Langlands correspondence is the following

$$\text{(TE) } \theta_{\widetilde{\pi}(\mu\varphi)} \text{ is a transfer of } \theta^i_\varphi.$$

As we said, this may be seen as a way to express θ^i_φ in terms of $\theta_{\widetilde{\pi}(\mu\varphi)}$. Since the latter is entirely determined (by LLC for general linear groups), condition (TE) entirely determines the characters θ^i_φ. Combined with condition (CE) this completely determines the local Langlands correspondence for unitary groups (if it exists).

Remark : All of these have been establish by C.P.Mok (following the work of Arthur) for quasisplit unitary groups. Namely Mok proved the existence of a local Langlands correspondence for these groups satisfying properties (STAB), (CE) and (TE).

This ends our discussion of LLC for unitary groups.

3.2 The Refined GGP Conjecture

3.2.1 The statement

As in the first lecture, it is more convenient to consider two GGP triples at the same time : (G_i, H_i, ξ_i) and (G_a, H_a, ξ_a) with $G_i = U(V_i) \times U(W_i)$ and $G_a = U(V_a) \times U(W_a)$ and where

- $d = dim(V_i) = dim(V_a)$ and $m = dim(W_i) = dim(W_a)$ are of distinct parities : $d \not\equiv m \ mod\ 2$;
- $V_i \not\simeq V_a$ and $W_i \not\simeq W_a$ (i.e. V_i and V_a are two representatives for the two isomophism classes of hermitian spaces of dimension d, and the same is true for W_i and W_a) ;

– G_i is quasisplit.

Let $\varphi \in \Phi_{temp}(d)$ and $\varphi' \in \Phi_{temp}(m)$ be two Langlands parameters. Gan, Gross and Prasad have defined two characters $\chi_{\varphi,\varphi'} : A_\varphi \to \{\pm 1\}$ and $\chi_{\varphi',\varphi} : A_{\varphi'} \to \{\pm 1\}$. We will recall their definition. For $s \in A_\varphi$ we will choose a lift of s, that is we choose a conjugate-dual form B of sign $(-1)^{d+1}$ on φ and lift s to an element of $Aut(\varphi, B)$. We will still denote by s the lift of s. Similarly, we choose for all $s' \in A_{\varphi'}$ a lift of s'. Now the characters $\chi_{\varphi,\varphi'}$ and $\chi_{\varphi',\varphi}$ are defined by

$$\chi_{\varphi,\varphi'}(s) = \epsilon(\varphi^{s=-1} \otimes \varphi', \psi)$$
$$\chi_{\varphi',\varphi}(s') = \epsilon(\varphi \otimes \varphi'^{s'=-1}, \psi)$$

for all $s \in A_\varphi$, $s' \in A_{\varphi'}$. Here, $\varphi^{s=-1}$ (resp. $\varphi'^{s'=-1}$) denotes the subrepresentation of φ where s acts by $-I$ (resp. the subrepresentation of φ' where s' acts by $-I$) and the ϵ factors are ϵ factors of Weil-Deligne representations. Gan, Gross and Prasad proved that these definitions don't depend on the choices of the lifts and that these are truly characters. We may now state the more refined version of GGP :

Theorem 1 *Assume the local Langlands correspondence for unitary groups exists and satisfies properties (STAB), (CE) and (TE). Then, for all $(\varphi, \varphi') \in \Phi_{temp}(d) \times \Phi_{temp}(m)$, there is an unique pair*

$$(\pi, \pi') \in \Pi_i(\varphi) \times \Pi_i(\varphi') \sqcup \Pi_a(\varphi) \times \Pi_a(\varphi')$$

such that $m(\pi \otimes \pi') = 1$ and moreover this pair is given by

$$\pi = \pi(\varphi, \chi_{\varphi,\varphi'})$$
$$\pi' = \pi(\varphi', \chi_{\varphi',\varphi})$$

3.2.2 The proof

The idea is fairly simple. On one hand, we have the formula for the multiplicity $m(\pi) = m_{geom}(\pi)$, $\pi \in Temp(G)$, where $G = U(V) \times U(W)$. This formula express the multiplicity in terms of the character θ_π of π. By using our hypothesis (CE) about classical endoscopy, we will be able to stabilize this formula i.e. to decompose it into a sum of some "stable" multiplicities living on endoscopic groups of G which are products of four quasisplit unitary groups. On the other hand, we have the formula for ϵ-factors $\epsilon(\pi_d \times \pi_m, \psi) = \epsilon_{geom}(\widetilde{\pi})\omega_{\pi_m}(-1)$, where $\pi_d \in Temp(M_d)$, $\pi_m \in Temp(M_m)$ are conjugate-dual and $\widetilde{\pi} \in Temp(\widetilde{M_d \times M_m})$ is the unique extension of $\pi = \pi_d \otimes \pi_m$ such that $\epsilon_{Whitt}(\widetilde{\pi}) = 1$. By the local Langlands correspondence for general linear groups, we have the equality $\epsilon(\pi_d \times \pi_m, \psi) = \epsilon(\varphi_d \otimes \varphi_m, \psi)$ where φ_d and φ_m are the Weil-Deligne representations corresponding to π_d and π_m respectively. So this formula may also be seen as a way to express the ϵ-factor $\epsilon(\varphi_d \otimes \varphi_m, \psi)$ in terms of the character $\theta_{\widetilde{\pi}}$ of $\widetilde{\pi}$. By our hypothesis (TE) about twisted endoscopy, we will be able to stabilize this formula i.e. decompose it as a sum of "stable" terms living on endoscopic groups of $\widetilde{G}$ that are products of two unitary groups. It will happen that these stable terms are equal to stable multiplicities. This will allow us to connect the two formulas and give an expression of $m(\pi)$ in terms of epsilon factors. This expression will give exactly the predictions of Gan, Gross and Prasad.

Let $(\varphi, \varphi') \in \Phi_{temp}(d) \times \Phi_{temp}(m)$. We would like to compute the multiplicity $m(\pi \otimes \pi')$ for all $(\pi, \pi') \in \Pi_i(\varphi) \times \Pi_i(\varphi') \sqcup \Pi_a(\varphi) \times \Pi_a(\varphi')$. By basic Fourier inversion, it is sufficient to compute

$$m(\varphi, s; \varphi', s') = \sum_{\chi \in \widehat{A}_\varphi, \chi' \in \widehat{A}_{\varphi'}} m\left(\pi(\varphi, \chi) \otimes \pi(\varphi', \chi')\right)$$

for all $s \in A_\varphi$ and $s' \in A_{\varphi'}$ and where we have set $m\left(\pi(\varphi, \chi) \otimes \pi(\varphi', \chi')\right) = 0$ if $\pi(\varphi, \chi) \otimes \pi(\varphi', \chi')$ is neither a representation of G_i nor of G_a.

Fix $s \in A_\varphi$ and $s' \in A_{\varphi'}$. From the formula for the multiplicity, we deduce an expression of $m(\varphi, s; \varphi', s')$ in terms of the chracters $\theta^i_{\varphi,s}$ and $\theta^a_{\varphi,s}$. Fix conjugate-dual forms B and B' on φ and φ' of sign $(-1)^{d+1}$ and $(-1)^{m+1}$ respectively. We may lift s and s' to elements of $Aut(\varphi, B)$ and $Aut(\varphi', B')$ that are involutions. Then we have decompositions

$$\varphi = \varphi^{s=1} \oplus \varphi^{s=-1}$$

$$\varphi' = \varphi'^{\,s'=1} \oplus \varphi'^{\,s'=-1}$$

Denote by d_+, d_-, m_+, m_- the dimensions of respectively $\varphi^{s=1}$, $\varphi^{s=-1}$, $\varphi'^{\,s'=1}$ and $\varphi'^{\,s'=-1}$. We may obviously find Langlands parameters $\varphi_+ \in \Phi_{temp}(d_+)$, $\varphi_- \in \Phi_{temp}(d_-)$, $\varphi'_+ \in \Phi_{temp}(m_+)$, $\varphi'_- \in \Phi_{temp}(m_-)$ and conjugate-dual characters μ_+, μ_-, μ'_+ and μ'_-, such that

$$\varphi^{s=1} = \mu_+\varphi_+$$

$$\varphi^{s=-1} = \mu_-\varphi_-$$

$$\varphi'^{\,s'=1} = \mu'_+\varphi'_+$$

$$\varphi'^{\,s'=-1} = \mu'_-\varphi'_-$$

Finally, consider four quasisplit hermitian spaces V_+, V_-, V'_+ and V'_- of respective dimensions d_+, d_-, m_+ and m_-. Using our hypothesis of classical endoscopy (CE), we may now stabilize the formula for $m(\varphi, s; \varphi', s')$ to express it in terms of the stable characters $\theta^i_{\varphi_+} \times \theta^i_{\varphi_-}$ and $\theta^i_{\varphi'_+} \times \theta^i_{\varphi'_-}$. As a result, we get something of the form

$$\text{(m-STAB)} \quad m(\varphi, s; \varphi', s') = m^{stab}_{\mu_+\mu'_-}(\varphi_+, \varphi'_-) m^{stab}_{\mu_-\mu'_+}(\varphi_-, \varphi'_-)$$

where for example $m^{stab}_{\mu_+\mu'_-}(\varphi_+, \varphi'_-)$ is an integral very similar to the one appearing in the formula for the multiplicity, where the function c_π is replaced by the product of the two functions $c^i_{\varphi_+} = \sum_{\pi\in\Pi^i(\varphi_+)} c_\pi$ and $c^i_{\varphi'_-} = \sum_{\pi\in\Pi^i(\varphi'_-)} c_\pi$.

On the other hand, we may also apply our twisted endoscopy assumption (TE) to stabilize $\epsilon_{geom}(\widetilde{\pi})$. Let $\underline{d}$ and $\underline{m}$ be two nonnegative integers of distinct parities. Consider two Langlands parameters $\underline{\varphi} \in \Phi_{temp}(\underline{d})$, $\underline{\varphi}' \in \Phi_{temp}(\underline{m})$ and two conjugate-dual characters μ and μ'. By twisting, we get two Langlands parameters $\mu\underline{\varphi} \in \widetilde{\Phi}_{temp}(\underline{d})$ and $\mu'\underline{\varphi}' \in \widetilde{\Phi}_{temp}(\underline{m})$. We will denote by $\widetilde{\pi}(\mu\underline{\varphi}) \in Temp(\widetilde{M}_{\underline{d}})$ and $\widetilde{\pi}(\mu'\underline{\varphi}') \in Temp(\widetilde{M}_{\underline{m}})$ the unique representations in the corresponding L-packets such that

$$\epsilon_{Whitt}(\widetilde{\pi}(\mu\underline{\varphi})) = \epsilon_{Whitt}(\widetilde{\pi}(\mu'\underline{\varphi}')) = 1$$

The formula for ϵ-factors gives us an equality

$$\epsilon(\mu\underline{\varphi} \otimes \mu'\underline{\varphi}', \psi) = \epsilon_{geom}(\widetilde{\pi})\omega_{\pi(\mu'\underline{\varphi}')}(-1)$$

where $\epsilon_{geom}(\widetilde{\pi})$ is an expression depending on the twisted character of $\widetilde{\pi} = \widetilde{\pi}(\mu\underline{\varphi}) \otimes \widetilde{\pi}(\mu'\underline{\varphi}')$. We may now use our hypothesis (TE) to stabilize this formula. The result is the following

$$(\epsilon\text{-STAB})\epsilon(\mu\underline{\varphi} \otimes \mu'\underline{\varphi}') = \begin{cases} m^{stab}_{\mu\mu'}(\underline{\varphi}, \underline{\varphi}') & \text{if } \mu\underline{\varphi} \otimes \mu'\underline{\varphi}' \text{ has sign } - \\ m^{stab}_{\mu}(\underline{\varphi}, 0)m^{stab}_{\mu'}(\underline{\varphi}', 0) & \text{if } \mu\underline{\varphi} \otimes \mu'\underline{\varphi}' \text{ has sign } + \end{cases}$$

Now, I claim that for all $\varphi \in \Phi_{temp}(n)$, we have $m_\mu(\varphi, 0) = c_\varphi(1)$. By the result of Rodier stated in the first lecture, there is an interpretation of $c_\varphi(1)$ in terms of Whittaker models. Namely, we have

$$c_\varphi(1) = \frac{\sharp\{\text{generic repns in } \Pi^i(\varphi)\}}{\sharp\{\text{conj classes of Whittaker data}\}}$$

Now (ϵ-STAB) gives us

$$\epsilon(\mu\underline{\varphi} \otimes \mu'\underline{\varphi}') = c_{\underline{\varphi}}(1)c_{\underline{\varphi}'}(1)$$

when $\mu\underline{\varphi} \otimes \mu'\underline{\varphi}'$ is of sign $+$. Using this formula for $\underline{\varphi}' = 0$, we get $c_{\underline{\varphi}}(1) = 1$ when $dim(\underline{\varphi})$ is odd. Using again this formula for $\underline{\varphi}'$ of odd dimension (so that $\underline{\varphi}$ is even-dimensional), we deduce that

$$c_{\underline{\varphi}}(1) = 1$$

for any $\underline{\varphi} \in \Phi_{temp}(n)$.

Now applying (m-STAB) to $s = s' = 0$, we get

$$m(\varphi, 0; \varphi', 0) = c_\varphi(1)c_{\varphi'}(1) = 1$$

This means exactly that there is an unique pair $(\pi, \pi') \in \Pi_i(\varphi) \times \Pi_i(\varphi') \sqcup \Pi_a(\varphi) \times \Pi_a(\varphi')$ such that

$$m(\pi \otimes \pi') = 1$$

As a consequence $(s, s') \mapsto m(\varphi, s; \varphi', s')$ is a character of $A_\varphi \times A_{\varphi'}$. Using again (m-STAB), we deduce that

$$(s, s') \mapsto m^{stab}_{\mu_+\mu'_-}(\varphi_+, \varphi'_-) m^{stab}_{\mu_-\mu'_+}(\varphi_-, \varphi'_+)$$

is also a character of $A_\varphi \times A_{\varphi'}$ (where $\varphi_+, \varphi_-, \varphi'_+, \varphi'_-$ and $\mu_+, \mu_-, \mu'_+, \mu'_-$ are constructed from s and s' as above). Combining this fact with the identity (ϵ-STAB). it is an easy exercise to show that (ϵ-STAB) remains true whatever the dimensions of $\underline{\varphi}$ and $\underline{\varphi}'$ are (there need not anymore be of distinct parities). Combining this with (m-STAB), we get the formula

$$m(\varphi, s; \varphi', s') = \epsilon(\mu_+\varphi_+ \otimes \mu'_-\varphi'_-, \psi)\epsilon(\mu_-\varphi_- \otimes \mu'_+\varphi'_+, \psi) = \epsilon(\varphi^{s=1} \otimes \varphi'^{s'=-1}, \psi)\epsilon(\varphi^{s=-1} \otimes \varphi'^{s'=1}, \psi)$$

Obviously, we have

$$\begin{aligned}\epsilon(\varphi^{s=1} \otimes \varphi'^{s'=-1}, \psi)\epsilon(\varphi^{s=-1} \otimes \varphi'^{s'=1}, \psi) &= \epsilon(\varphi \otimes \varphi'^{s'=-1}, \psi)\epsilon(\varphi^{s=-1} \otimes \varphi', \psi)\epsilon(\varphi^{s=-1} \otimes \varphi'^{s'=-1}, \psi)^{-2} \\ &= \chi_{\varphi,\varphi'}(s)\chi_{\varphi',\varphi}(s')\epsilon(\varphi^{s=-1} \otimes \varphi'^{s'=-1}, \psi)^{-2}\end{aligned}$$

Combined with the fact that $\epsilon(\varphi^{s=-1} \otimes \varphi'^{s'=-1}, \psi)^2 = 1$, this gives the desired result. ■